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AFWAL-TR-85-3002

THE METHOD OF RESIDUAL MINIMIZATION
IN COMPRESSIBLE STEADY FLOWS

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April 1985
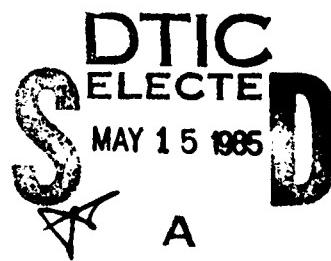
Interim Report for period March 1983 - June 1984

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FLIGHT DYNAMICS LABORATORY
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REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION	1b. RESTRICTIVE MARKINGS
Unclassified	None

2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	Approved for public release, Distribution Unlimited

4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S)
UDR-TR-84-46	AFWAL-TR-85-3002

6a. NAME OF PERFORMING ORGANIZATION	6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION
University of Dayton Research Institute		Air Force Wright Aeronautical Laboratories Flight Dynamics Laboratory (AFWAL/FIBRC)

6c. ADDRESS (City, State and ZIP Code)	7b. ADDRESS (City, State and ZIP Code)
300 College Park Avenue Dayton, Ohio 45469	Wright-Patterson AFB, OH 45433-6553

8a. NAME OF FUNDING/SPONSORING ORGANIZATION	8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
Flight Dynamics Laboratory	AFWAL/FIBRC	F33615-83-K-3207

8c. ADDRESS (City, State and ZIP Code)	10. SOURCE OF FUNDING NOS.			
	PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
Wright-Patterson Air Force, Ohio 45433	61102F	2304	N1	10

11. TITLE (Include Security Classification) The Method of Residual Minimization in Compressible Steady Flows (Uncl)

12. PERSONAL AUTHOR(S)
Karl G. Guderley

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
Interim Report	FROM Mar 83 TO June 84	April 1985	248

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB. GR.	Steady compressible flows; Far field conditions; Wakes; Euler equations of variational problems.
01	01		
20	04		

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Theoretical aspects of residual minimization in transonic flows are discussed for different formulations of the basic equations. By a special choice of the metric in function space, one de-emphasizes short wave errors, which are technically less important but cause slow convergence. Far field conditions are included in the approach by considering the distant (noncomputed) part of the flow field as a superelement characterized by shape functions which solve the linearized equations. The matching of the three dimensions between the distant flow field and the computed part poses a difficult problem. A simple example shows a rather strong propagation of matching errors. Wake capturing can be accomplished even if one retains in nearly all of the flow field the idea of a potential flow. Linearized examples for supersonic problems give some insight in the questions of stability and accuracy.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT	21. ABSTRACT SECURITY CLASSIFICATION
UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☒ DTIC USERS ☐	UNCLASSIFIED

22a. NAME OF RESPONSIBLE INDIVIDUAL	22b. TELEPHONE NUMBER (Include Area Code)	22c. OFFICE SYMBOL
Charles L. Keller	513-255-7384	AFWAL/FIBRC

DD FORM 1473, 83 APR   EDITION OF 1 JAN 73 IS OBSOLETE.

PREFACE

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INTRODUCTION

Because of their nonlinearities all computations of compressible flows must proceed iteratively. The question of convergence then plays an important role. It is true in present day methods the convergence difficulties have been overcome, usually by special ingenious methods like upwinding or density modification. In methods based on residual minimization convergence is guaranteed from the outset. The converged solution is then approached in a manner quite different from, say, a finite difference approach. The modifications of an existing approximation in a method of residual minimization are of a global character while in other approaches one carries out rather localized modifications (line relaxation or alternate direction methods at best). So one can regard residual minimization as a genuine alternative. This fact makes a closer analysis desirable.

Which of the approaches, finite differences, finite elements or residual minimization, is most practical can, of course, be decided only by numerical experimentation. In this regard the present report makes no contribution; it provides a theoretical clarification of many other aspects.

The idea of residual minimization has been proposed in the past. In an unsophisticated form it has, however, the disadvantage that the residual functional is overly sensitive against short wave error, while the elimination of long wave errors (which are more detrimental from a technical point of view) is rather slow. A first step towards an improvement has been made by Johnson, James, and their co-workers (Reference 1) by the use of an alternative form of the basic differential equation. For the usual formulation in terms of a velocity potential the idea has come to fruition in the work of Bristeau and her co-workers (Reference 2). The present report takes a somewhat more general point of view. As far as Reference 2 is concerned, the present analysis adds very little to the actual procedure. The author

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hopes, neverthe-less, that for some of the readers the motivation given here will be welcome.

The idea of residual minimization will be applied to three different formulations of the flow problem. One in terms of the velocity components, a second one in terms of the dependent variables chosen by Johnson and James (the potential and a vector potential, or in two dimensions the potential and a the stream function), and the third one in terms of the variables used by Bristeau, et alii (the potential and an auxiliary scalar function).

The essential elements of the methods are:

1. The characterization of the overall residual by a non-negative functional. The form of the functional depends upon the formulation chosen. The flow problem is solved when the Euler equations of the variational problems are satisfied.

2. *The Euler equations are solved indirectly by a search* procedure in the space of approximations to the flow field. (If the flow field is represented by means of the velocity vector, then an element of the function space is given by a vector valued function of the space coordinates (x,y,z).) For a general element of the function space the flow equations will not be satisfied; but the element found by the minimization process will represent the velocity field. In a flow field described by the velocity potential, the element of the function space is a scalar function of x, y, and z. The search procedure will be carried out as a conjugate gradient method. To define the gradient in the function space one needs a definition for the distance between two elements. The choice of the distance definition is crucial in making the procedure less sensitive against short wave errors. With obvious modifications the distance definition proposed in Reference 2 can be carried over to other formulations.

The basic concepts are discussed in Section II for an overly simple example. The formulation of the actual problem in terms of

either the velocity vector, the variables of Johnson and James (potential and a vector potential), or the variables of Bristeau (potential and an auxiliary scalar function) is developed in Sections III, IV, and V. In each case the form of the functional that is to be minimized and the distance definition in the function space are give. Some insight into the working of the method is obtained by considering the Euler equations of the variational problem (which in practice are solved by the search procedure). They can be interpreted as partial differential equations for the residuals in the original equations. The partial differential equations are closely related to the partial differential equations for the flow field (although they have different dependend variables). The procedure yields in addition boundary conditions for the residuals, they are of a nature which guarantees in conjunction with the Euler equations that the residuals will be identically equal to zero throughout the flow field. Accordingly one always has a combination of two sets of boundary conditions, those of the original problem and those for the residuals. The discussions show the need for additional conditions along a shock (actually they arise automatically).

The functionals that are to be minimized are chosen in a manner which anticipates the Prandtl-Glauert coordinate transformation. In a subsonic field this leads to a quick convergence in the distant field. Although residual minimization is not practical in supersonic problems, the method is discussed for such cases because of two reasons.

1. The method of residual minimization would be of questionable value in transonic problems if it should fail in purely supersonic problems.

2. It is of some interest to see whether the method can be carried over to time dependent problems; a purely supersonic problem can be regarded as a simple prototype.

In the variables of Johnson and James, the boundary conditions for the auxiliary vector potential are of somewhat unorthodox nature, because they involve derivatives of the same

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order as those that occur in the partial differential equations.
Besides, the vector potential is not uniquely determined. The
boundary conditions can be reformulated and auxiliary conditions
can be introduced which make the vector potential unique. A
detailed discussion is included in Section IV. Also discussed in
these sections is the treatment of shocks, either by shock
capturing or by shock identification.

Two-dimensional flow fields with circulations or three-
dimensional flow fields with a wake require special modifications.
Section VI gives neuristic discussions regarding the nature of
certain free parameters which must be properly chosen and the
special role of the Kutta condition.

Euler codes are frequently used to capture wakes as well as
shocks. In transonic flow the entropy change in the shock is
negligible. The whole flow field, except for wake is therefore
irrotational. Section VII gives an overview of the equation for
steady, isentropic, isoenergetic flows. The flow differential
equations can be expressed solely in terms of the velocity vector.
It is, therefore, possible to modify the functional that is to be
minimiz⁻¹ in such a way that wake capturing becomes possible. Of
course, no claim of novelty can be made to this review of the
basic equations. It has been carried out to give a clear picture
of the relation between different formulations.

At the outer edge of the computed part of the flow field one
must impose boundary conditions of some form. For initial
numerical experiments it is probably sufficient if one takes
examples for a body within a wind tunnel. Then there is no doubt
about the conditions that are to be imposed. But actually one is
concerned with the flow in free air. It is suggested that the
distant field (i.e. the field outside of the computed part of the
flow field) be regarded as a superelement with a finite number of
shape functions which satisfy the flow equations (actually in
their linearized form). The coefficients of the particular
solutions are the shape parameters. In the distant field there
are no residuals, but variations of the shape parameters affect

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the computed part of the flow field because of the matching between computed and distant field. Section VIII derives formulae for particular solutions in the distant field.

Two- and three-dimensional flows with circulation differ in so far as in two dimensions no identification of the wake is necessary and also because in two dimensions the contribution of the circulation in the distant field can be expressed by an exact solution of the linearized equations. In three dimensions one must introduce in the distant field an approximation for the position of the wake and approximations for the effect of the vorticity distribution within the wake on the outer boundary of the computed flow field. Sections IX and X discuss far field conditions and circulation effects for two- and three-dimensional problems. A simple example shown in Section X gives some insight into the effect of a mismatch between the wake in the computed part of the flow field and in the distant field. While (in subsonic flow) local errors die out exponentially with distance, there are errors in the wake region generated at the far boundary which die out only linearly.

In Section XI the concept of density modification (which is suggested as a means of stabilizing the flow field in the supersonic region) is studied in an idealized form. In the interpretation given here density modification introduces in the flow differential equations a derivative in the flow direction which is one order higher than the derivatives of the original formulation. A simple example demonstrates that density modification gives some damping to particular solutions which are present without this measure, but in addition it introduces a new type of particular solutions. As one proceeds downstream, particular solutions of this type decrease very fast in a subsonic region and increase very fast in a supersonic region. In a one-dimensional example of the flow thro. a Laval nozzle one recognizes that a solution of this type causes the velocity to break away rather suddenly from a supersonic basic solution toward smaller velocities until subsonic velocities are reached, and then the curve attaches itself to a subsonic flow which is very close

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to one without density modification. Here this type of particular solution yields a smoothed out form of a shock. The Section includes some remarks about an other condition by which one can discriminate against expansion shocks.

The last Sections consider the working of the method in discretized form for linearized supersonic flows. Considered are the Bristeau formulation and an approach in terms of the variables of Johnson and James (here ϕ and ψ). The discussions are the counterpart of a von Neumann analysis, but because residual minimization is a global concept one must consider the flow field as a whole. As always, one studies the behavior of particular solutions. In the ϕ,ψ method the boundary conditions for the Euler equations of the variational problem in the exit cross section are important. The method of Bristeau is found to be dispersive (even if one does not apply special measures as density modification). Under certain circumstances it may become unstable (even though the concept of residual minimization guarantees convergence). The ϕ,ψ method is always stable, it introduces amplitude and phase errors simultaneously. Both methods are sensitive against a nonalignment of the grid with respect to the velocity direction. The results show that a rather fine grid is necessary to give an acceptable accuracy.

During these investigations the author found that one must guard rather carefully against carrying over intuitive concepts that have evolved in other approaches to the method of residual minimization. This, and not an inherent complexity of the methods, is one reason for the length of the present report.

SECTION II

AN ILLUSTRATIVE EXAMPLE

The purpose of this section is to bring the essential ideas
into focus by means of a simple example. We consider the
linearized potential equation for compressible flow

$$(1-M^2)\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \tag{1}$$

(M is the free stream Mach number, ϕ the velocity potential x,y,z
refers to a Cartesian system of coordinates in which the x axis
is aligned with the free stream direction.) We restrict our
attention to the error incurred in different approximations for
ϕ. Then it suffices, if one considers homogeneous boundary
conditions. In the example to be discussed we confine ourselves
to regions bounded by planes x = const, y = const, and z = const.
At the bounding surfaces we prescribe ϕ = 0. Alternatively one
can regard the expressions to be discussed as the terms of a
Fourier development in the infinite space. The present
discussions is facilitated by the fact that the operator in
Eq. (1) commutes with the Laplace operator. Now let ϕ be the
error in some approximation to the solution (which would be
defined by inhomogeneous boundary conditions). The residual is
then given by the left side of Eq. (1). Minimizing the residual,
one postulates

$$J = \int_{\Omega} [(1-M^2)\phi_{xx} + \phi_{yy} + \phi_{zz}]^2 d\tau = \text{Min} \tag{2}$$

where Ω is the region considered and $d\tau$ = dxdydz is the volume
element. The error ϕ can be represented by a linear combination
of eigenfunctions of the operator defined by the left side of
Eq. (1)

$$\phi = \sum_i c_i \sin(\alpha_i x)\sin(\beta_i y)\sin(\gamma_i z) \tag{3}$$

The constants α_i, β_i, and γ_i are chosen so that the boundary conditions prescribed at the planes x = const, y = const, z = const are satisfied. Because of the orthogonality of the individual terms, one then obtains for the integral J, (Eq. (2)) except for a normalization constant

$$J = \sum_i c_i^2 [(1-M^2)\alpha_i^2 + \beta_i^2 + \gamma_i^2]^2 \qquad (4)$$

The minimization gives, of course, $c_i = 0$. One has to minimize a quadratic function of the c_i's (here it appears in the simplest possible form). For large α_i, β_i, or γ_i (that is for short wave errors) the coefficient of the c_i are very large. Geometrically speaking, one has to find the center of extremely elongated ellipsoids in the c_i space. Even if one truncates the sum a search process will converge extremely slowly.

We define the scalar product of two vectors $\vec{c}$ and $\vec{d}$ (with components c_i and d_i) in the c_i space by

$$[\vec{c},\vec{d}] = \sum_i c_i d_i \qquad (5)$$

and the gradient vector in the c_i space by $\vec{g}$ with components g_i. Then one finds from Eq. (4)

$$g_i = 2c_i [(1-M^2)\alpha_i^2 + \beta_i^2 + \gamma_i^2]^2$$

The equivalent of the gradient is a function in the x,y,z space given by

$$g(x,y,z) =$$

$$\sum_i 2c_i [(1-M^2)\alpha_i^2 + \beta_i^2 + \gamma_i^2]^2 \sin(\alpha_i x)\sin(\beta_i y)\sin(\gamma_i z) \qquad (6)$$

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If the original function ϕ is twice differentiable, then the residual

$r(x,y,z,) =$

$$\sum_i c_i [(1-M^2)\alpha_i^2 + \beta_i^2 + \gamma_i^2] \sin(\alpha_i x)\sin(\beta_i y)\sin(\gamma_i z)$$

will exist. This, however, is not sufficient to ensure the existence of the gradient function $g(x,y,z)$ because there one encounters the square of the bracket, except of course if one truncates the infinite series which amounts to an artificially smoothing.

Part of the remedy proposed by Bristeau, et al., is the introduction of an auxiliary function ξ, defined by

$$\Delta\xi = \text{residual} = (1-M^2)\phi_{xx} + \phi_{yy} + \phi_{zz} \qquad (7)$$

(Here Δ is the Laplace operator). Imposing suitable homogeneous boundary conditions for ξ, one finds that $\xi = 0$ if the residual vanishes. One obtains here

$$\xi = \sum_i c_i \frac{(1-M^2)\alpha_i^2 + \beta_i^2 + \gamma_i^2}{\alpha_i^2 + \beta_i^2 + \gamma_i^2} \sin(\alpha_i x)\sin(\beta_i y)\sin(\gamma_i z)$$

The function ξ converges in the same manner as ϕ. One uses ξ to define a different functional which is minimized if the residual vanished. At this stage one has several options. One can for instance choose

$$J = \int_\Omega \xi^2 d\tau = \text{Min} \qquad (8)$$

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or

$$J = \int_{\Omega} \text{grad}\,\xi \cdot \text{grad}\,\xi\,d\tau = \text{Min} \qquad (9)$$

This gives, except for a constant factor respectively

$$\sum_i c_i^2 \left[\frac{(1-M^2)\alpha_i^2 + \beta_i^2 + \gamma_i^2}{\alpha_i^2 + \beta_i^2 + \gamma_i^2} \right]^2 = \text{Min} \qquad (10)$$

or

$$\sum_i c_i^2 \frac{[(1-M^2)\alpha_i^2 + \beta_i^2 + \gamma_i^2]^2}{\alpha_i^2 + \beta_i^2 + \gamma_i^2} = \text{Min} \qquad (11)$$

Each option leads to $c_i = 0$ for all values of i, but the
formulations differ in the manner in which they assign weight to
the different components of the error. The first formulation
gives nearly the same weights to particular solutions (with equal
amplitude) for errors in ϕ, the second one gives nearly equal
weight for errors in the velocity grad ϕ, independent of the
waviness of the particular solutions. Physically the second
alternative might appear preferable, although long wave errors
have a larger cumulative effect.

An alternative formulation is best explained for the two-
dimensional problem. The Cauchy Riemann equations are the
equivalent of the Laplace equation. Introducing velocity
components u and v for, respectively, the x and y directions, one
has (in analogy to the Cauchy Riemann equations)

$$(1-M^2)u_x + v_y = 0$$
$$u_y - v_x = 0 \qquad (12)$$

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Equations of the same type (but with different boundary conditions) are obtained by introducing a potential and a stream function. One then obtains as an alternative to Eq. (1) in two dimensions

$$(1-M^2)\phi_x - \psi_y = 0$$

$$\phi_y + \psi_x = 0 \tag{13}$$

The use of a potential and a stream function (or in three dimension of a vector potential) has been proposed by Johnson, et al. One notices that in order to obtain the velocity components in the formulation Eq. (12) as well as in Eq. (1) one needs a differentiation. To obtain the same accuracy with Eq. (12) on one hand and Eqs. (13) and (1) on the other hand, one needs in the second case either more sophisticated shape functions or smaller elements.

For solving Eq. (12) one now postulates

$$J = \int_\Omega \{[(1-M^2)u_x + v_y]^2 + [u_y - v_x]^2\} d\tau = \text{Min} \tag{14}$$

Setting

$$u = \sum_i c_i \cos(\alpha_i x)\sin(\beta_i y)$$

$$v = \sum_i d_i \sin(\alpha_i x)\cos(\beta_i y) \tag{15}$$

one obtains

$$(1-M^2)u_x + v_y = \sum_i [-(1-M^2)c_i\alpha_i + d_i\beta_i]\sin(\alpha_i x)\sin(\beta_i y)$$

$$u_y - v_x = \sum_i [c_i\beta_i - d_i\alpha_i]\cos(\alpha_i x)\cos(\beta_i y)$$

Then

$$J = \sum_i [-(1-M^2)c_i\alpha_i + d_i\beta_i]^2 + [c_i\beta_i - d_i\alpha_i]^2 \qquad (16)$$

This formulation procedure has the advantage that one does not need to solve the Laplace equation in order to evaluate the function that is to be minimized. In the formulation of Eq. (2) which leads to Eq. (4) the α_i and β_i occur in the fourth power, here as well as in the formulation which arises from Eq. (9) one encounters as dominant for large α_i and β_i only second powers. At least for the formulation in terms of ϕ and ψ one again gives about equal weight to errors in the velocities pertaining to different particular solutions . In the formulation in terms of the velocity components u and v one obtains greater weight on short wave particular solutions for the errors in the velocity components. But at least one does not encounter ellipsoids which are quite as extremely elongated.

The ultimate aim of each of these procedures is the solution of the Euler equation of the variational problem. The technique to be applied is some search procedure (not a direct solution); of direct interest is not the value of the minimum (which is close to zero in any case) but its location. Because we are dealing with expressions for the errors, the minimum will lie in the present examples at $c_i = 0$ for all i. The technique proposed for the search procedure is the method of conjugate gradients. The definition of a gradient presupposes the definition of the distance between two elements of the function space in question. In the above approaches such elements are given by $\phi(x,y,z)$, or u(x,y) and v(x,y), or $\phi(x,y)$ and $\psi(x,y)$. The distance is implied by the definition of a scalar product. Notice that the definition of the distance has no effect on the location of the minimum, it has, however, an effect on the manner in which one approaches the minimum by iteration. A definition

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of the distance which might appear natural has been given above. Bristeau and her co-workers propose a more sophisticated definition of the scalar product.

Let ϕ_1 and ϕ_2 be two elements of the ϕ space. The definition of the scalar product analogous to that in Eq. (5) is

$$[\phi_1, \phi_2]_0 = \int_\Omega \phi_1(x,y,z)\phi_2(x,y,z)d\tau \qquad (17)$$

Bristeau uses instead

$$[\phi_1, \phi_2]_1 = \int_\Omega \text{grad}\,\phi_1 \cdot \text{grad}\,\phi_2 d\tau \qquad (18)$$

To illustrate the idea further one might also discuss

$$[\phi_1, \phi_2]_2 = \int \Delta\phi_1 \Delta\phi_2 d\tau \qquad (19)$$

A thorough mathematical justification is found in the original paper of Bristeau, et al. Here we restrict ourself to the manipulative part. Using the formulation (Eq. (2)) of the functional, one obtains from Eq. (4) for the variation δJ of J

$$\delta J = \sum_i 2c_i \delta c_i [(1-M^2)\alpha_i^2 + \beta_i^2 + \gamma_i^2]^2 \qquad (20)$$

To find the function which is the analogue to the gradient vector, we write

$$\delta J = [g, \delta\phi]$$

As before, we write

$$g = \sum_i d_i \sin(\alpha_i x)\sin(\beta_i y)\sin(\gamma_i z) \qquad (21)$$

One always has

$$\delta\phi = \sum_i \delta c_i \sin(\alpha_i x)\sin(\beta_i y)\sin(\gamma_i z)$$

Using the scalar product $[\ ]_2$, Eq. (19), one obtains (except for a normalization constant),

$$[g, \delta\phi]_2 = \sum_i d_i \delta c_i [\alpha_i^2 + \beta_i^2 + \gamma_i^2]^2 \qquad (22)$$

Hence by comparing with Eq. (20)

$$d_i = 2c_i \frac{[(1-M^2)\alpha_i^2 + \beta_i^2 + \gamma_i^2]^2}{[\alpha_i^2 + \beta_i^2 + \gamma_i^2]^2} \qquad (23)$$

One obtains a gradient function which has the same convergence characteristics as the original function ϕ. The drawback of this definition of the scalar product lies in the fact, that one must solve a bi-harmonic equation in order to determine g. This is seen in the following manner. One has to solve

$$\int_\Omega \Delta g \, \delta \Delta\phi \, d\tau = 2 \int_\Omega [(1-M^2)\phi_{xx} + \phi_{yy} + \phi_{zz}][(1-M^2)\delta\phi_{xx} + \delta\phi_{yy} + \delta\phi_{zz}]$$

Carrying out integrations by part on the left and on the right (and omitting, for simplicity, the terms which arise at the boundaries) one obtains

$$\int_\Omega \Delta \Delta g \, \delta\phi \, d\tau = 2 \int_\Omega [(1-M^2)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}][(1-M^2)\phi_{xx} + \phi_{yy} + \phi_{zz}]\delta\phi \, d\tau$$

Hence

$$\Delta \Delta g = 2[(1-M^2)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}][(1-M^2)\phi_{xx} + \phi_{yy} + \phi_{zz}].$$

The gradient vector must be determined at the point of the function space to which one has been led by the preceding

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iteration step. This is a specific function ϕ. The right-hand side is, therefore, known. The gradient function g is, therefore, determined from an inhomogeneous bi-harmonic equation.

The use of the scalar product Eqs. (17) and (18) in conjunction with the definition (Eq. (2)) for the functional that is to be minimized gives a gradient vector that does not belong to the space of admissible functions ϕ.

Bristeau actually minimizes the functional Eq. (9) and uses (for the search procedure) the scalar product Eq. (18). The variation of J is then obtained from Eq. (16)

$$\delta J = 2 \sum_i c_i \delta c_i \frac{[(1-M_i^2)\alpha_i^2 + \beta_i^2 + \gamma_i^2]^2}{\alpha_i^2 + \beta_i^2 + \gamma_i^2}$$

With the definition of Eq. (21), the gradient vector is obtained

$$[g, \delta\phi]_1 = \sum_i d_i \delta c_i (\alpha_i^2 + \beta_i^2 + \gamma_i^2)$$

By comparing these two equations one obtains

$$d_i = 2c_i \frac{[(1-M_i^2)\alpha_i^2 + \beta_i^2 + \gamma_i^2]^2}{[\alpha_i^2 + \beta_i^2 + \gamma_i^2]^2} \tag{24}$$

The gradient vector is the same for the two approaches (at least in this example). In the method just described one must solve Poisson equations twice, once to determine ζ, the second time to determine the gradient vector. In a conjugate gradient method one carries out one-dimensional searches, during this process one must evaluate the functional J and, therefore, also the function ζ several times. Each evaluation of ζ requires one application of a Poisson solver.

The solution of the bi-harmonic equation is roughly equivalent to two applications of a Poisson solver. In the method described first it is not necessary for the determination

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of the functional J, which is encountered n the search procedure, that one solve partial differential equations.

In an approach based on the formulation Eq. (12) (or (13)) of the basic equations one has elements in the u,v space defined by

$$u = \sum_i c_i \cos(\alpha_i x) \sin(\beta_i y)$$

$$v = \sum_i d_i \sin(\alpha_i x) \cos(\beta_i y)$$

The functional to be minimized is given in Eq. (16). Then

$$\delta J = 2 \sum_i \delta c_i \{-(1-M^2)\alpha_i[-(1-M^2)c_i\alpha_i + d_i\beta_i] + \beta_i[c_i\beta_i - d_i\alpha_i]\}$$

$$+ 2 \sum_i \delta d_i \{\beta_i[-(1-M^2)c_i\alpha_i + d_i\beta_i] - \alpha_i[c_i\beta_i - d_i\alpha_i]\}$$

The scalar product in the u,v space is defined by

$$[(u,v),(g_1,g_2)] = \int_\Omega (\text{grad}u \cdot \text{grad}g_1 + \text{grad}v \cdot \text{grad}g_2)d\tau$$

We set

$$g_1(x,y) = \sum_i g_{1,i}\cos(\alpha_i x)\sin(\beta_i y)$$

$$g_2(x,y) = \sum_i g_{2,i}\sin(\alpha_i x)\cos(\beta_i y)$$

Then (with the definitions (Eq. (15)), except for a normalization constant

$$[(u,v),(g_1,g_2)] = \sum_i c_i g_{1,i}(\alpha_i^2 + \beta_i^2) + \sum_i d_i g_{2,i}(\alpha_i^2 + \beta_i^2)$$

and

$$[\delta(u,v),(g_1,g_2)] = \sum_i \delta c_i g_{1,i}(\alpha_i^2 + \beta_i^2) + \sum_i \delta d_i g_{2,i}(\alpha_i^2 + \beta_i^2)$$

Hence

$$g_{1,i} = 2 \frac{-(1-M^2)\alpha_i[-(1-M^2)c_i\alpha_i + d_i\beta_i] + \beta_i[c_i\beta_i - d_i\alpha_i]}{(\alpha_i^2 + \beta_i^2)}$$

$$g_{2,i} = 2 \frac{\beta_i[-(1-M^2)c_i\alpha_i + d_i\beta_i] - \alpha_i[c_i\beta_i - d_i\alpha_i]}{(\alpha_i^2 + \beta_i^2)}$$

The powers of α_i and β_i in the numerators and denominators of these expressions are the same, the components of the gradient vector g_1 and g_2 converge therefore in the same manner as the velocity components u and v. In other words, the gradient vector lies within the u,v space.

By using in this approach the $[\ ,\ ]_1$ definition involving the gradients in the u,v space, one considers distances which correspond to high frequency waviness errors as large in comparison to distances in the direction of low waviness errors. This compensates for the fact that in the definition of J high waviness contributions (in the velocity) have more weight than low waviness contributions; the hyperellipsoids are, therefore, closer to hyperspheres, which facilitates the search procedure.

If the Mach number is 0, then all procedures lead to exact spheres, the method of steepest descend (which is the first step in a conjugate gradient method) reaches the minimum in one step.

Because of the shortcuts which are possible in this simple example these discussions are somewhat deceptive; but they show the guiding ideas. One should be aware of the fact that the definition of the functional to be minimized, and the distance definition for the function space are separate entities.

SECTION III

RESIDUAL MINIMIZATION FOR STEADY COMPRESSIBLE
FLOWS TREATED IN TERMS OF THE VELOCITY COMPONENTS

The basic equations are given by

$$\text{div}(\rho \vec{w}) = 0 \qquad (25)$$

$$\text{curl } \vec{w} = 0 \qquad (26)$$

where $\vec{w}$ is the velocity vector and ρ the density. We assume that ρ and $\vec{w}$ are made dimensionless, respectively, with density and velocity at sonic speed. Then one has for a gas with constant ratio γ of the specific heats

$$\rho = [((\gamma+1)/2) - ((\gamma-1)/2)\vec{w}^2]^{1/(\gamma-1)} \qquad (27)$$

It might seem as if Eqs. (25) and (26) represent four equations for $\vec{w}$. One will, however, remember that the individual components of Eq. (26) are connected to each other by

$$\text{div curl } \vec{w} = 0.$$

The boundary conditions are familiar from the formulation in terms of the potential. Let Γ be the boundary of the region Ω in which the problem is to be solved. One prescribes the potential along a portion Γ_1 of Γ, and the mass flow normal to the boundary along a portion Γ_2. In subsonic flows Γ_1 and Γ_2 do not overlap and their union gives Γ. In supersonic flows Γ_1 and Γ_2 overlap in the entrance cross section and their union does not include the exit cross section.

Let $\vec{m} = \rho \vec{w}$ be the mass flow vector. One has along Γ_2

$$\vec{m} \cdot \vec{e}_n - f(x,y,z) = 0 \qquad x,y,z \in \Gamma_2 \qquad (28)$$

where $\vec{e}_n$ is the unit vector in the direction of the normal to Γ.

If the potential ϕ is given along Γ_1, one can evaluate grad $\phi \times \vec{e}_n$ and thus obtains the boundary condition

$$\vec{w} \times \vec{e}_n - \text{grad } \phi \times \vec{e}_n = 0 \qquad x,y,z \in \Gamma_1 \qquad (29)$$

The boundary condition expressed on Γ_1 in this form guarantee that the velocity components given within the surface Γ_1 are compatible with Eq. (26).

The boundary conditions Eq. (28) and (29) do not always determine the solution uniquely. Consider, for instance, a subsonic parallel flow through a channel with constant cross section. The side walls belong to Γ_2, there one has obviously $\vec{m} \cdot \vec{e}_n = 0$. A well determined flow field is obtained if one prescribes $\vec{m} \cdot \vec{e}_n = \text{const}$ in the entrance cross section, (where the constant is determined by the Mach number) and $\phi = \text{const}$ in the exit cross section. Then, according to Eq. (29), $\vec{w} \times \vec{e}_n = 0$ in the exit cross section.

At least mathematically, one may impose the following alternative boundary conditions: $\phi = c_1$ in the entrance cross section, $\phi = c_2$ in the exit cross section, where the difference $c_2 - c_1$ is determined by the Mach number and the length of the tunnel. Here it is obviously not sufficient if one sets $\vec{w} \times \vec{e}_n = 0$ in the entrance and the exit cross sections, for these conditions are compatible with a parallel flow of any Mach number. To attain uniqueness one may add the requirement that the difference of the potential between the entrance and the exit cross section has an assigned value

$$\int_A^B \vec{w} \cdot d\vec{s} = c_2 - c_1 \qquad (29a)$$

In this formulation the difference of the potential is expressed by the integral along some path connecting a point A of the entrance cross section with a point B of the exit cross section. Physically more natural are conditions which (in the

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present example) require that the potential in the entrance and in the exit cross section be constant, but leaves these constants undetermined. This yields the requirement $\vec{w} \times \vec{e}_n = 0$, the velocity vector is normal to these surfaces. Instead of the potential difference one prescribes the total mass flow through the entrance cross section.

$$\int_{\Gamma_{1,1}} \vec{m} \cdot \vec{e}_n \, do - c = 0 \qquad (29b)$$

Here $\Gamma_{1,1}$ is the entrance cross section. Because $\vec{m}$ is a nonlinear function of $\vec{w}$, this is a nonlinear condition. In all practical cases such supplementary conditions can be found by inspection.

We make the following observation of a theoretical nature. Eqs. (25) and (26) define an operator operating on $\vec{w}$. The usual definition of an operator includes the domain of the operator, the domain is given by all functions which satisfy the boundary conditions (Eqs. (28) and (29)). But $\vec{m}$ is a nonlinear function of $\vec{w}$. The domain of the operator, therefore, depends upon the functions on which it operates. The domain defines the space of functions admissible in the minimization process. In this problem this is not necessarily a linear space. If $\vec{w}^{(1)}$ is an admissible function, then $\vec{w}^{(1)}$ multiplied by a constant is not necessarily an admissible function, because it might not satisfy Eq. (28). (This observation does not apply, if $f(x,y,z) \equiv 0$.) This has the following practical consequence. The process of residual minimization is carried out by a sequence of linear searches. For a function $\vec{w}$ given by

$$\vec{w} = \vec{w}^{(1)} + a\vec{w}^{(2)}$$

One forms the functional that is to be minimized, a is the parameter to be varied during the search, $\vec{w}^{(1)}$ is an admissible function, so is the whole expression $\vec{w}$ for a small of first

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order. But $\vec{w}$ need not be an admissible function for α not infinitesimally small.

The functional to be minimized is given by

$$J_1(\vec{w}) = (1/2)\int_\Omega [(\operatorname{div} \vec{m})^2 + \operatorname{curl} \vec{w} \cdot \operatorname{curl} \vec{w}]d\tau \qquad (30)$$

where $d\tau$ is the volume element in the space of Cartesian coordinates x, y, z. We shall denote the components of $\vec{w}$ in the x, y, and z directions respectively by subscripts 1, 2, and 3. Subscripts x, y, and z are reserved for partial derivatives. In this formulation $\vec{w}$ is subject to the boundary conditions, Eqs. (28) and (29). This formulation is practical only if the function f occuring in Eq. (28) is zero and if an additional condition of the type Eq. (29a) or (29b) is not needed.

The space of admissible functions can be extended to a linear space if one includes the boundary conditions in the definition of the functional that is to be minimized.

$$J_2(\vec{w}) = (1/2 \int_\Omega [(\operatorname{div} \vec{m})^2 + \operatorname{curl} \vec{w} \cdot \operatorname{curl} \vec{w}]d\tau$$

$$+ (c_1/2)\int_{\Gamma_1} [(\vec{w} - \operatorname{grad} \phi) \times \vec{e}_n]^2 do \qquad (31)$$

$$+ (c_2/2)\int_{\Gamma_2} [\vec{m} \cdot \vec{e}_n - f]^2 do$$

$$+ (c_3/2)(\int_{\Gamma_{11}} \vec{m} \cdot \vec{e}_n do - c)do)^2$$

do denotes the scalar surface element. In this formulation no boundary conditions are imposed on $\vec{w}$.

In the discretized form of the problem this formulation balances the failure to satisfy the boundary conditions exactly against the failure to satisfy the partial differential

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equations. It is interesting to note that in the Bateman minimum formulation for subsonic flows such a balancing for the boundary condition Eq. (28) (the mass flow through the boundary) appears automatically. The Bateman formulation cannot accommodate a similar balancing for the potential.

The mass flow vector $\vec{m}$ is a function of $\vec{w}$. Let

$$\lim_{\epsilon \to 0} \epsilon^{-1}[\vec{m}(\vec{w}+\epsilon\vec{v}) - \vec{m}(\vec{w})] = M\vec{v} \tag{32}$$

M is an operator operating on the vector valued function $\vec{v}$. M depends upon $\vec{w}$. It can be written in the form of a symmetric matrix

$$M\vec{v} = \rho \begin{bmatrix} 1 - w_1^2/a^2 & -w_1 w_2/a^2 & -w_1 w_3/a^2 \\ -w_1 w_2/a^2 & 1 - w_2^2/a^2 & -w_2 w_3/a^2 \\ -w_1 w_3/a^2 & -w_2 w_3/a^2 & 1 - w_3^2/a^2 \end{bmatrix} \vec{v} \tag{33}$$

with

$$a^2 = ((\gamma+1)/2) - ((\gamma-1)/2)\vec{w}^2 \quad . \tag{34}$$

Varying $\vec{w}$ in Eq. (31) one obtains

$$\delta J_2 = \int_\Omega \{div\ \vec{m}\ div(\delta\vec{m}) + curl\ \vec{w} \cdot curl(\delta\vec{w})\}d\tau$$

$$+ c_1 \int_{\Gamma_1} [(\vec{w} - grad\ \phi) \times \vec{e}_n] \cdot [\delta\vec{w} \times \vec{e}_n]do \tag{35}$$

$$+ c_2 \int_{\Gamma_2} [\vec{m} \cdot \vec{e}_n - f][\delta\vec{m} \cdot \vec{e}_n]do$$

$$+ c_3 (\int_{\Gamma_{11}} \vec{m} \cdot \vec{e}_n do - c) \int_{\Gamma_{11}} \delta\vec{m} \cdot \vec{e}_n do$$

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Because of Eq. (32) one has

$$\delta\vec{m} = M \delta\vec{w}$$

No boundary conditions are imposed on $\delta\vec{w}$. In principle, Eq. (35) is the starting point for the numerical evaluation of the gradient vector in the $\vec{w}$ space for tne discretized form of the problem. We derive the Euler equations of the variational problem. Some observations about the numerical aspects are made at the end of this section. The following discussion is carried out to provide some theoretical insight. In the numerical work many of the equations which appear in this discussion will be bypassed.

To transform the integrals one uses the following relations

$$\int_{\Omega} a \ \mathrm{div} \ \vec{b} \ d\tau = \int_{\Gamma} a(\vec{b}\cdot\vec{e}_n)do - \int_{\Omega} \vec{b}\cdot\mathrm{grad} \ a \ d\tau \qquad (36)$$

$$\int_{\Omega} \vec{c}\cdot\mathrm{curl} \ \vec{d} \ d\tau = -\int_{\Gamma} [\vec{c},\vec{d},\vec{e}_n]do + \int_{\Omega} \mathrm{curl} \ \vec{c}\cdot\vec{d} \ d\tau \qquad (37)$$

Here

$$[\vec{c},\vec{d},\vec{e}_n] = (\vec{c}\times\vec{d})\cdot\vec{e}_n$$

is the mixed product. We note the familiar relations

$$[\vec{c},\vec{d},\vec{e}_n] = [\vec{e}_n,\vec{c},\vec{d}] = [\vec{d},\vec{e}_n,\vec{c}] = -[\vec{e}_n,\vec{d},\vec{c}] \qquad (38)$$

Identifying in Eq. (35) div $\vec{m}$ and $\delta\vec{m} = M\delta\vec{w}$ with, respectively $\vec{a}$ and $\vec{b}$ in Eq. (36), and curl $\vec{w}$ and $\delta\vec{w}$ with respectively $\vec{c}$ and $\vec{d}$ in Eq. (37), applying one of the relations (Eq. (38)) and using the fact that the matrix M is symmetric

$$\delta J_2 = \int_\Omega [-M \text{ grad div } \vec{m} + \text{curl curl } \vec{w}] \cdot \delta\vec{w} \, d\tau$$

$$+ \int_\Gamma [\text{div } \vec{m} \, M\vec{e}_n - \vec{e}_n \times \text{curl } \vec{w}] \cdot \delta\vec{w} \, do$$

$$+ c_1 \int_{\Gamma_1} [(\vec{w} - \text{grad } \phi) \times \vec{e}_n] \cdot [\delta\vec{w} \times \vec{e}_n] do \qquad (39)$$

$$+ c_2 \int_{\Gamma_2} [\vec{m} \cdot \vec{e}_n - f][M\vec{e}_n \cdot \delta w] do$$

$$+ c_3 (\int_{\Gamma_{11}} \vec{m} \cdot \vec{e}_n \, do - c) \int_{\Gamma_{11}} M\vec{e}_n \cdot \delta\vec{w} \, do$$

The corresponding expression which arises from the minimum formulation in which one assumes that the admissible functions satisfy the original boundary conditions, Eqs. (28) and (29), and that Eq. (29a or b) need not be imposed, is obtained by setting $c_1 = c_2 = 0$. In addition, one has then

$$\delta\vec{w} \times \vec{e}_n = 0 \text{ on } \Gamma_1, \qquad \text{(because of Eq. (29))}$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (40)$

$$\delta\vec{m} \cdot \vec{e}_n = M\vec{e}_n \cdot \delta\vec{w} = 0 \text{ on } \Gamma_2, \quad \text{(because of Eq. (28))}$$

One then obtains

$$\delta J_1 = \int_\Omega [-M \text{ grad div } \vec{m} + \text{curl curl } \vec{w}] \cdot \delta\vec{w} \, d\tau$$

$$+ \int_{\Gamma - \Gamma_2} (\text{div } \vec{m}) M\vec{e}_n \cdot \delta\vec{w} \, d_o - \int_{\Gamma - \Gamma_1} [\vec{e}_n, \text{curl } \vec{w}, \delta\vec{w}] do \qquad (41)$$

The restriction of the boundary integrals to $\Gamma - \Gamma_2$ and $\Gamma - \Gamma_1$ occurs because of the boundary conditions, Eq. (40).

Eq. (41) yields the Euler equations

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$$-M \ \text{grad}(\text{div } \vec{w}) + \text{curl curl } \vec{w} = 0 \qquad (42)$$

with the original boundary conditions for $\vec{w}$, Eqs. (28) and (29), and additional boundary conditions

$$\text{div } \vec{m} = 0 \qquad \text{on } \Gamma - \Gamma_2$$

$$\text{curl } \vec{w} \times \vec{e}_n = 0 \qquad \text{on } \Gamma - \Gamma_1 \qquad (43)$$

The following interpretation has some theoretical interest. Let r_1 and $\vec{r}_2$ be the residuals in the original Eqs. (25) and (26)

$$r_1 = \text{div } \vec{m}$$

$$\vec{r}_2 = \text{curl } \vec{w} \qquad (44)$$

The Euler equations (Eq. (42)) with the boundary conditions (Eq. (43)) then assume the form

$$-M \ \text{grad } r_1 + \text{curl } \vec{r}_2 = 0 \qquad (45)$$

$$r_1 = 0 \qquad \text{on } \Gamma - \Gamma_2$$

$$\vec{r}_2 \times \vec{e}_n = 0 \qquad \text{on } \Gamma - \Gamma_2 \qquad (46)$$

It follows from the second of Eqs. (44), that

$$\text{div } \vec{r}_2 = 0 \qquad (47)$$

Eqs. (45) and (47) form a system of four partial differential equations for the four residuals r_1 and $\vec{r}_2$, and Eqs. (46) give the pertinent boundary conditions. The operator in this system is in essence the adjoint operator of the linearized system of Eqs. (25) and (26). One can examine this system and its boundary conditions to confirm whether it guarantees that $r_1 = 0$ and $\vec{r}_2 = 0$. This interpretation shows the role of the boundary

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conditions, Eq. (46). A more detailed discussion will be carried out for the other forms of the minimization hypothesis.

The formulations Eqs. (30) or (31) are not applicable if shocks are present in the flow field, even if one interprets the integrals in the sense of generalized functions. (So far no provisions have been made to exclude expansion shocks. At the moment we refer to expansion as well as to compression shocks.) At a shock the velocity vector will jump. Unless the shock conditions are satisfied, the expressions $\text{div } \vec{m} = \text{div } \rho\vec{w}$ and $\text{curl } \vec{w}$ will give δ functions. The integrand of Eqs. (30) or (31) will then contain the square of δ functions. But such terms make no sense. The same conclusion can be drawn from Eq. (35). It can be considered as a weighted residual expression with weights given by $\text{div } \rho\vec{m}$ and $\text{curl } \delta\vec{w}$. At a shock these weights are δ functions , but the residuals to which the weights are applied are δ functions too.

In the discretized form of the equations this difficulty is somewhat camouflaged, for there one will approximate discontinuous functions by functions which are continuous. But the difficulty remains. At the shock the weights would become unduely large and this creates difficulties in a search procedure. This becomes more and more pronounced as the mesh is refined. It would mean a severe limitation of the method of residual minimization if this difficulty cannot be surmounted. For instance, no extension to Euler codes (in which one works with the Euler equations of motion rather than the idea of a potential flow) would be possible.

The difficulty can be overcome, if one excludes, the shock by a contour closely surrounding it and includes the shock conditions in the minimum formulation. (Of course, as long as one introduces the idea of irrotationality (Eq. (26)) one cannot satisfy the shock conditions completely.) With the assumptions that the shocks are rather weak it suffices if one postulates that the mass flow normal to the shock upstream and downstream of

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it and the velocity components tangential to the shock be the same.

Let the shock surface be given by

$$F_s(x,y,z) = 0 \qquad (48)$$

In most cases one can write

$$x - f_s(y,z) = 0 \qquad (49)$$

A unit vector normal to the shock is then given by

$$\vec{e}_{ns} = \text{grad } F_s / |\text{grad } F_s| \qquad (50)$$

Let $[...]_-^+$ denote the difference of the argument of this symbol between the upstream and downstream side of the shock. The shock conditions then assume the form

$$[\vec{m}]_-^+ \cdot \vec{e}_{ns} = 0$$

$$[\vec{w}]_-^+ \times \vec{e}_{ns} = 0 \qquad (51)$$

To express the shock conditions in the minimum formulation one must add to the expression Eqs. (30) or (31) the terms

$$(1/2)\int_{r_s} c_3([\vec{m}]_-^+ \cdot \vec{e}_{ns})^2 do + (1/2\int_{r_s} c_4([w]_-^+ \times \vec{e}_{ns})^2 do \qquad (52)$$

r_s is the upstream side of the shock surface, do its (scalar) element, $c_3 > 0$, $c_4 > 0$, c_3 and c_4 need not be constant. Now to be varied is the function $\vec{w}(x,y,z)$ considered as different on the upstream and downstream side of the shock and also the shock surface (given by the functions $f_s(x,y)$).

Next we explore some aspects of the search procedure by the method of conjugate gradients. The gradients of J_1 or J_2 must be formed in a function space whose elements are given by $\vec{w}$ and f_s

(Eq. (49)). Let us first disregard the presence of shocks. The definition of a gradient presupposes the existence of a distance definition in the functions space. Such a definition is implied by the scalar product. An unsophisticated definition of the scalar product of two functions $\vec{u}$ and $\vec{v}$ of the $\vec{w}$ space is given by

$$[\vec{u},\vec{v}] = \int_\Omega \vec{u} \cdot \vec{v} \, d\tau \qquad (53)$$

We identify $\vec{u}$ with the gradient $\vec{g}$ and $\vec{v}$ with $\delta\vec{w}$. The gradient vector is then obtained from

$$\int_\Omega \vec{g} \cdot \delta\vec{w} \, d\tau = \delta J \qquad (54)$$

If one uses the formulation, Eq. (30), the gradient vector is subject to the boundary conditions

$$\vec{g} \times \vec{e}_n = 0 \qquad \text{on } \Gamma_1$$
$$\qquad\qquad\qquad\qquad\qquad (55)$$
$$M\vec{g} \cdot \vec{e}_n = 0 \qquad \text{on } \Gamma_2$$

One obtains by substituting the expressions Eq. (39) or (41) for δJ into Eq. (54)

$$\vec{g} = -M \, \text{grad div } \vec{m} + \text{curl curl } \vec{w} \qquad (56)$$

With this formulation the gradient function $\vec{g}$ arises from the current $\vec{w}$ by a process which includes two differentiations. If the $\vec{w}$ space is restricted to twice differentiable functions, then $\vec{g}$ will not have this property. Moreover, the function $\vec{g}$ so obtained will in general not satisfy the boundary conditions (Eq. (51)).

One sees, that this choice of the scalar product does not give a legitimate procedure, at least from a theoretical point of view. The difficulty seems to vanish in the discretized version

of the problem, for then one operates in a finite dimensional vector space. The difficulty is caused by the fact, already mentioned in Section II, that short wave errors are given undue weight in the functional J. The method itself is attractive because one does not need to solve partial differential equations in order to determine the gradient function $\vec{g}$. The question arises whether a multigrid procedure, in which one uses the coarse grid to eliminate most of the long wave errors, and then proceeds to finer grids for which (temporarily at least) one is only confronted with short wave errors will provide an effective way out of this difficulty. This question can be decided only by numerical experimentation.

The difficulty just described vanishes, if one defines in analogy to the procedure of Bristeau, et al.

$$[\vec{u},\vec{v}] = \sum_{i=1}^{3} (\int_{\Omega} (\text{grad } u_i \cdot \text{grad } v_i) d\tau + \int_{\Gamma} c_5 u_i v_i do) \qquad (57)$$

$$c_5 \geq 0, \quad c_5 \text{ need not be constant}$$

One will remember that according to definitions made above the subscripts 1, 2, and 3 refer to components of the vector in the x, y, and z directions, respectively. In this definition the scalar product is written in terms of components of the vectors $\vec{u}$ and $\vec{v}$, but one convinces oneself easily that the expression remains the same if one transforms it to another system of Cartesian coordinates. (This can also be shown by means of tensor calculus.)

If $c_5 = 0$, then functions $\vec{u}$ and $\vec{u} + \vec{c}$ (where $\vec{c}$ is a constant vector field) will give the same scalar product. Usually this possibility is excluded by boundary conditions. Proceeding in the same manner as above, one obtains

$$[\vec{g}, \delta\vec{w}] = \sum_{i=1}^{3} (\int_{\Omega} (\text{grad } g_i \cdot \text{grad } \delta\vec{w}_i) d\tau + \int c_5 g_i \delta\vec{w}_i do$$

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and by applying Eq. (36) to this equation, with a = δw_i and b = grad g_i

$$[\vec{g}, \delta\vec{w}] = \sum_{i=1}^{3} (-\int_{\Omega} \text{div grad } g_i \delta\vec{w}_i d\tau + \int_{\Gamma} [\text{grad } g_i \cdot \vec{e}_n + c_5 g_i] \delta\vec{w}_i do)$$

This is slightly rewritten and set equal to δJ

$$\delta J = -\int_{\Omega} \Delta\vec{g} \cdot \delta\vec{w} + \sum_{i=1}^{3} \int_{\Gamma} (\text{grad } g_i \cdot \vec{e}_n) \delta w_i do + \int_{\Gamma} c_5 g \cdot \delta\vec{w} do \qquad (58)$$

Comparing with Eq. (39) or (41), one obtains immediately

$$\Delta\vec{g} = M \text{ grad div } \vec{m} - \text{curl curl } \vec{w} \qquad (59)$$

This is a Poisson equation for each of the three components of $\vec{g}$. One also obtains a set of boundary conditions for each of the components. The simplest form arises, if one uses Eq. (39). Then one has

$$\text{grad } g_i \cdot \vec{e}_n + c_5 g_i = (\text{div } \vec{m} + c_2(\vec{m} \cdot \vec{e}_n - f))(M\vec{e}_n)_i$$

$$\qquad (60)$$

$$+ \{[\text{curl } \vec{w} + c_1(\vec{w} - \text{grad } \phi)] \times \vec{e}_n\}_i \text{ on } \Gamma;$$

$$+ c_3(\int_{\Gamma_{11}} \vec{m} \cdot \vec{e}_n do - c)(M\vec{e}_n)_i$$

$c_1 = 0$ on Γ_2, $c_2 = 0$ on Γ_1, the last term is present on Γ_{11} only.

As before, the subscripts i refers to the individual components of the vectors. One obtains indeed a well posed problem for each of the components of $\vec{g}$.

Only the right-hand side of these equations changes, if one determines the gradient function $\vec{g}$ for different values of $\vec{w}$. If one has fixed elements (and this presupposes that no shocks are present), this results in considerable simplications in the

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repeated solution of the Poisson equation. Even the introduction of special elements in the vicinity of the shock is not too detrimental, because only a few of the elements in the grid system need to be changed.

The results are less symmetric, if one postulates that the admissible functions satsify the boundary conditions of the original problem. It is assumed that Eqs. (29a) or (29b) are not applicable. Using the expression Eq. (41) for δJ, one has because of Eqs. (28) and (29)

$$\delta \vec{w} \times \vec{e}_n = 0$$
$$\vec{g} \times \vec{e}_n = 0$$
on Γ_1 \hfill (61)

$$M \delta \vec{w} \cdot \vec{e}_n = 0$$
$$M \vec{g} \cdot \vec{e}_n = 0$$
on Γ_2 \hfill (62)

Consider first the portion $\Gamma_1 - (\Gamma_1 \cap \Gamma_2)$ of the boundary. The exclusion of the intersection of Γ_1 and Γ_2 is necessary in supersonic flow; in subsonic flows Γ_1 and Γ_2 do not overlap. One obtains because of the second of Eqs. (61)

$$\vec{g} = \sim \vec{e}_n$$

Denoting by $\vec{e}_{n_1}$, $\vec{e}_{n_2}$, and $\vec{e}_{n_3}$ the three components of $\vec{e}_n$, one obtains

$$g_1 = |\vec{g}| e_{n_1} , \quad g_2 = |\vec{g}| e_{n_2} , \quad g_3 = |\vec{g}| e_{n_3} \tag{63}$$

This amounts to two relations between g_1, g_2, and g_3. Furthermore, by comparing Eq. (58) with Eq. (41) (observing that $\Gamma - \Gamma_1 \cup \Gamma_2$ belongs to $\Gamma - \Gamma_2$)

$$\sum_i (\text{grad } g_i \cdot \vec{e}_n) \delta w_i = (\text{div } \vec{m})(M\vec{e}_n) \cdot \delta \vec{w}$$

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One has because of the first of Eqs. (61)

$$\delta \vec{w} = \text{const } \vec{e}_n$$

$$\delta w_i = \text{const } e_{n_i}$$

Therefore,

$$\sum_i (\text{grad } g_i \cdot \vec{e}_n) e_{n_i} = (\text{div } \vec{m})(M\vec{e}_n) \cdot \vec{e}_n \qquad (64)$$

This is the third condition along $\Gamma_1 - \Gamma_1 \quad \Gamma_2$.

An alternative to Eqs. (63) is obtained as follows. Denoting by $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ unit vectors in respectively the x, y, and z directions it follows from the second of Eqs. (61), that

$$[\vec{e}_n, \vec{g}, \vec{e}_1] = 0$$

$$[\vec{e}_n, \vec{g}, \vec{e}_2] = 0 \qquad (65)$$

and

$$[\vec{e}_n, \vec{g}, \vec{e}_3] = 0$$

One has

$$e_{n_1}[\vec{e}_n, \vec{g}, \vec{e}_1] + e_{n_2}[\vec{e}_n, \vec{g}, \vec{e}_2] + e_{n_3}[\vec{e}_n, \vec{g}, \vec{e}_3] = 0$$

because

$$\vec{e}_n = e_{n_1}\vec{e}_1 + e_{n_2}\vec{e}_2 + e_{n_3}\vec{e}_3$$

The expression Eqs. (65) are, therefore, linearly dependent.

On $\Gamma_2 - (\Gamma_2 \quad \Gamma_1)$ one has the following situation. Comparing Eq. (58) with Eq. (41) one obtains

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$$\sum_i [\text{grad } g_i \cdot \vec{e}_n] + [\vec{e}_n, \text{curl } \vec{w}, \vec{e}_i] \delta w_i = 0 \tag{66}$$

Here we have written

$$[\vec{e}_n, \text{curl } \vec{w}, \delta\vec{w}] = [\vec{e}_n, \text{curl } \vec{w}, \sum_i \vec{e}_i \delta w_i]$$

Eq. (66) is subject to the restriction given by the first of Eqs. (62)

$$(M\vec{e}_n) \cdot \delta\vec{w} = 0 \tag{67}$$

The combination of Eqs. (66) and (67) gives two homogeneous equations of the form

$$a_1 \delta w_1 + a_2 \delta w_2 + a_3 \delta w_3 = 0$$

$$b_1 \delta w_1 + b_2 \delta w_2 + b_3 \delta w_3 = 0$$

One seeks conditions which guarantee that the first equation is satisfied whenever the second equation is satisfied. The second equation can for instance be satisfied by the choice of δw_1 and δw_2 while $\delta w_3 = 0$. Then one finds

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0 \tag{68a}$$

In a similar manner one obtains

$$\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = 0 \tag{68b}$$

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and

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = 0 \qquad (68c)$$

The three conditions are linearly dependent. They amount to two conditions which are linear in grad $g_i \cdot \vec{e}_n$. In addition, one has from the second in Eq. (62)

$$M\vec{g} \cdot \vec{e}_n = 0 \qquad (69)$$

In supersonic flows the entrance cross section is the union of Γ_1 and Γ_2. There one has simply from the second equation in Eqs. (61) and (62)

$$g_i = 0, \quad i = 1,2,3 \qquad (70)$$

The exit cross section is given by $\Gamma - \Gamma_1 \cup \Gamma_2$ which belongs to $\Gamma - \Gamma_1$ and $\Gamma - \Gamma_2$. There is no restriction on $\delta\vec{w}$. Both surface integrals in Eq. (41) must therefore be taken into account. Then one obtains

$$\text{grad } g_i \cdot \vec{e}_n = \{(\text{div } \vec{m})M\vec{e}_n + \text{curl } \vec{w} \times \vec{e}_n\}_i \qquad (71)$$

where the subscript i refers (as always) to the ith component.

These discussions show that one obtains in every case (even in a supersonic flow) well-posed boundary value problems for Poisson equations for the individual components of the gradient function $\vec{g}$. If the approximation to the boundary conditions is not included in the minimization process, then the individual components of $\vec{g}$ are connected to each other by some of the boundary conditions. The steps which one carried out in practice are actually less complicated than the theory developed here.

In the presence of shocks one must consider the variations of $\vec{w}$ upstream and downstream of the shock as independent and take

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the variation of the shock shape into account. The details especially for the determination of the gradient function $\vec{g}$ are best worked out in the discretized form. Nevertheless, we write down the additional terms which occur in δJ in order to provide some theoretical insight.

Let $\Gamma_s{}^+$ and Γ_s^- be the two banks of the shock surface, respectively, upstream and downstream of the shock. They have, of course, a common normal, except that an outer normal vector to Γ_s^+ is the inner normal vector to Γ_s^-. Let $\vec{e}_{ns}$ be the outer normal to Γ_s^+. A vector in the direction of $\vec{e}_{ns}$ is obtained from Eq. (49)

$$\vec{e}_{ns} = (1+f_{s,x}^2 + f_{s,y}^2)^{1/2}[\vec{e}_1 - f_{s,y}\vec{e}_2 - f_{s,z}\vec{e}_3] \tag{72}$$

As defined above, $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ are unit vectors in the direction of the x, y, and z axes, respectively. Subscripts y and z refer to derivatives.

We consider contributions to the variation of J (Eqs. (30) or (31)), due to the presence of a shock. At Γ_s c_1 and c_2 zero. The effect of the surface terms is immediately seen from Eq. (41)

$$\delta J_{corr}^{(1)} = \int_{\Gamma_s^+} [(\text{div }\vec{m})M\vec{e}_{ns} + \text{curl }\vec{w} \times \vec{e}_{ns}] \cdot \delta w^+ do$$

$$- \int_{\Gamma_s^-} [(\text{div }\vec{m})M\vec{e}_{ns} + \text{curl }\vec{w} \times \vec{e}_{ns}] \cdot \delta w^- do$$

The + or − superscript on $\delta\vec{w}$ is superfluous, it is a reminder that upstream and downstream of the shock $\delta\vec{w}$ need not be the same. A second correction comes from the fact that the integrand of J may jump at the shock. If the shock moves downstream by δf_s, then the upstream region is enlarged and the downstream region made smaller. One obtains a second correction

$$\delta J_{corr}^{(2)} = 1/2 \int_{\Gamma_s^+} [(\text{div }\vec{m})^2 + (\text{curl }\vec{w})^2]_-^+ \delta f_s(x,y) dx dy$$

Further terms occur because of the expression Eq. (52) which in the presence of shocks must be added to the functional J. Using the expression Eq. (72) for $\vec{e}_{ns}$ one obtains for the projection of the surface element do_s on a plane $x = $ const

$$dydz = do(1 + f_{sy}^2 + f_{sz}^2)^{-1/2}$$

It is practical to substitute Eq. (72) into Eq. (52) and to use this equation in the form

$$(1/2) \int_{shock} \{\tilde{c}_3(y,z)([\vec{m}]_-^+ \cdot (\vec{e}_1 - f_{sy}\vec{e}_2 - f_{sz}\vec{e}_3))^2$$

$$+ \tilde{c}_4(y,z)([\vec{w}]_-^+ \times (\vec{e}_1 - f_{sy}\vec{e}_2 - f_{sz}\vec{e}_3))^2\}dydz \tag{73}$$

where

$$\tilde{c}_i(y,z) = c_i(y,z)(1 + f_{sy}^2 + f_{sz}^2)^{1/2} \qquad i = 3,4$$

are considered as fixed positive functions of y and z defined within the projection of the shock on a plane $x = $ const.

One then obtains, because of the variation of $\vec{w}$

$$\delta J_{corr}^{(3)} =$$

$$\int_{\Gamma_s} \{\tilde{c}_3(y,z)([\vec{m}]_-^+ \cdot (\vec{e}_1 - f_{sy}\vec{e}_2 - f_{sz}\vec{e}_3))((M^+\delta w^+ - \bar{M}\delta w^-) \cdot (\vec{e}_1 - f_{sy}\vec{e}_2 - f_{sz}\vec{e}_3)$$

$$+ \tilde{c}_4(y,z)([\vec{w}]_-^+ \times (\vec{e}_1 - f_{sy}\vec{e}_2 - f_{sz}\vec{e}_3)) \cdot ([\delta\vec{w}]_-^+ \times (\vec{e}_1 - f_{sy}\vec{e}_2 - f_{sz}\vec{e}))\}dydz \tag{74}$$

Furthermore, because of the variation of f_s

$$\delta J_{corr}^{(4)} = \int_{\Gamma_s} \{\tilde{c}_3(y,z)([\vec{m}]_-^+ \cdot (\vec{e}_1 - f_{sy}\vec{e}_2 - f_{sz}\vec{e}_3))([\vec{m}]_-^+ \cdot (-\delta f_{sy}\vec{e}_2 - \delta f_z\vec{e}_3))$$

$$+ \tilde{c}_4(y,z)([\vec{w}]_-^+ \times (\vec{e}_1 - f_{sy}\vec{e}_2 - f_{sz}\vec{e}_3)) \cdot ([\vec{w}]_-^+ \times (-\delta f_{sy}\vec{e}_2 - \delta f_{sz}\vec{e}_3))\}dydz \tag{75}$$

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Notice that in the last expression variations of the derivatives f_{sy} and f_{sz} are encountered.

Since an element of the function space is now formed by $\vec{w}(x,y,z)$ combined with $f_s(y,z)$, the gradient function $\vec{g}$ must now include aside from $\vec{g}$, a term $\vec{g}_s(y,z)$, which is associated with $f_s(y,z)$. The same holds for the scalar product in the function space. If $f_s^{(1)}$ and $f_s^{(2)}$ are two functions f_s, then one adds a term

$$c_6 \int_{\Gamma_s} f_s^{(1)} f_s^{(2)} dydz$$

to the scalar product. The occurrence of derivatives of f_s in Eq. (73) has again the effect that short waves in the function f_s occur with undue weight. Theoretically at least, one might, therefore, prefer to choose a scalar product

$$c_6 \int_{\Gamma_s} (\text{grad } f_s^{(1)} \cdot \text{grad } f_s^{(2)} dydz$$

where the gradient is formed in the y,z plane. Practical experience must show, whether or not this complication can be avoided in the discretized form of the problem.

In describing the discretization process we assume that $w(x,y,z)$ and $f_s(y,z)$ is approximated in a manner familiar from finite element approaches, namely as a linear combination of shape functions $\vec{\phi}_j$ (of finite support) with coefficients p_j and $p_{s,j}$ which are called shape parameters.

$$\vec{w}(x,y,z) = \sum_j p_j \vec{\phi}_j(x,y,z,\ldots p_{s,k}\ldots) \tag{76}$$

$$f_s(y,z) = \sum_k p_{s,k} \psi_k(x,y,z) \tag{77}$$

In the presence of shocks some elements and, therefore, also the pertinent shape functions will depend upon the shock shape. This is indicated by the arguments $p_{s,k}$ in $\vec{\phi}_j$. The shape functions

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for the $\vec{w}$ are written as vectors. Actually one will introduce shape functions separately for each component. We identify the components by first subscripts 1,2,3. One will use the same shape function for each component.

$$w_i(x,y,z) = \sum_j p_{i,j} \phi_j(x,y,z,\ldots p_{s,k}\ldots) \quad i = 1,2,3 \qquad (78)$$

In situations where the formulation Eq. (30) can be applied, some shape parameters are determined by the boundary conditions and not subject to variations. An approximation to the flow field is fully described by the shape parameters $p_{i,j}$ and $p_{s,j}$. The basic subroutine is one which computes J for a given set of shape parameters.

$$J = J(\ldots p_{1,j}\ldots,\ldots p_{2,j}\ldots,\ldots p_{3,j}\ldots,\ldots p_{s,k}\ldots)$$

The variation of J is then expressed by

$$\delta J = \sum_{i=1}^{3} \sum_j \frac{\partial J}{\partial p_{i,j}} \delta p_{i,j} + \sum_j \frac{\partial J}{\partial p_{s,j}} \delta p_{s,k} \qquad (79)$$

The derivatives are probably best determined by numerical differentiation. Writing momentarily $p_{i,k}$ for $p_{i,k}$ as well $p_{s,k}$ one has

$$\partial J/\partial p_{i,k} = \epsilon^{-1}[J(\ldots.p_{i,k}+\epsilon..) - J(\ldots p_{i,k}..)] \qquad (80)$$

Arguments not listed remain unchanged. This requires the repeated evaluation of J. But the amount of work is less than it might appear, because the support of the shape functions ϕ_j and ψ_j is finite; the integrations need only to be carried out over a rather limited part of the x,y,z space and of the shock. Eq. (79) can be written as the scalar product of one row and one column matrix. Let

$$[\ldots\delta p\ldots] = [\ldots\delta p_{1,j}\ldots,\ldots\delta p_{2,j}\ldots,\ldots\delta p_{3,j}\ldots,\ldots\delta p_{s,j}..] \qquad (81)$$

and

$$[...\partial J/\partial p...]$$

$$= [..\partial J/\partial p_{1,j}..,...\partial J/\partial p_{2,j}..,...\partial J/\partial p_{3,j}..,\partial J/\partial p_{s,j}..] \tag{82}$$

Then

$$\delta J = [...\delta p...][...\partial J/\partial p...]^{\dagger} \tag{83}$$

Let $\vec{g}(x,y,z)$ with components $g_i(x,y,z), i=1,2,3$ be the gradient functions pertaining to $\vec{w}$, and $g_s(y,z)$ the gradient functions pertaining to f_s. We denote the shape parameters pertaining to g_i and g_s respectively by $q_{i,j}$ and $q_{s,j}$. Then

$$g_i(x,y,z) = \sum_j q_{i,j}\phi_j(x,y,z,...p_{s,k}...) \tag{84}$$

$$g_s(y,z) = \sum_j q_{s,j}\psi_j(y,z) \tag{85}$$

Eqs. (77) and (78) give

$$\delta w_i = \sum_j \delta p_{i,j}\phi_j(x,y,z...p_{s,k}...) + \sum_j p_{i,j}\sum_k(\partial\phi j/\partial p_{s,k})\delta p_{s,k} \tag{86}$$

$$\delta f_s = \sum_j \delta p_{s,j}\psi_s(y,z) \tag{87}$$

We first determine the gradient vector for the definition Eq. (53) of the scalar product. One obtains

$$[g_i,\delta w_i] = \int_{\Omega} g_i \delta w_i d\tau \tag{88}$$

$$= \int_{\Omega}(\sum_{j1} q_{i,j1}\phi_{j1})(\sum_{j2} \delta p_{i,j2}\phi_{j2} + \sum_{j2} p_{i,j2}\sum_k(\partial\phi_{j2}/\partial p_{s,k})\delta p_{s,k})\delta\tau$$

and

$$\int_{\Gamma_s} c_6 g_s \delta f_s dydz = \int_{\Gamma_s} c_6 \sum_{j1} q_{s,j1}\psi_{j1}\sum_{j2}\delta p_{s,j2}\psi_{j2}dydz$$

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We have omitted the arguments of the snape functions. In the shape functions $\vec{\phi}_j$ the arguments $p_{s,k}$ are encountered. In the factor $q_{i,j1}\phi_{j1}$ in Eq. (88) the variation of the $p_{s,k}$ need not be taken into account; it would appear in an expression for the second variation.

These expressions can now be written in the form

$$\int_\Omega g_i \delta w_i d\tau = \sum_{j1} \sum_{j2} M^{(11)}_{j2,j1} q_{i,j1} \delta p_{i,j2} + \sum_{j1} \sum_{\kappa} M^{(i,21)}_{k,j1} q_{i,j1} \delta p_{s,\kappa} \qquad (89)$$

and

$$\int_{\Gamma_3} c_6 g_s \delta f_s dydz = \sum_{j1} \sum_{j2} M^{(22)}_{j2,j1} q_{s,j1} \delta p_{s,j2} \qquad (90)$$

where

$$M^{(11)}_{j2,j1} = M^{(11)}_{j1,j2} = \int_\Omega \phi_{j1}\phi_{j2} d\tau \qquad (91)$$

$$M^{(i,21)}_{k,j1} = \int_\Omega \phi_{j1} \sum_{j2} p_{i,j2}(\partial\phi_{j2}/\partial p_{s,\kappa}) d\tau \qquad (92)$$

$$M^{(22)}_{j1,j2} = \int_\Omega c_6 \psi_{j1}\psi_{j2} dydz \qquad (93)$$

Let

$$[\ldots q\ldots] = [\ldots q_{1,j}\ldots,\ldots q_{2,j}\ldots,\ldots q_{3,j}\ldots,\ldots q_{s,j}\ldots] \qquad (94)$$

Then with the definitions of Eqs. (81) and (94)

$$\sum_{i\Omega}\int g_i \delta w_i + \int_{\Gamma_s} c_6 g_s \delta f_s = [\ldots\delta p\ldots]M^{(2)}[\ldots q\ldots]^+ \qquad (95)$$

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where $M^{(2)}$ is a matrix, partitioned as follows

$$
M^{(2)} = \begin{bmatrix}
M^{(11)} & 0 & 0 & 0 \\
0 & M^{(11)} & 0 & 0 \\
0 & 0 & M^{(11)} & 0 \\
M^{(1,21)} & M^{(2,21)} & M^{(3,21)} & M^{(22)}
\end{bmatrix}
\tag{96}
$$

Equating the right-hand sides of Eq. (95) and Eq. (83) and observing that the resulting equation must be correct for any choice of the δp's one obtains the system of equations

$$
M^{(2)} \begin{bmatrix}
\vdots \\
q_{1,m1} \\
\vdots \\
q_{2,m2} \\
\vdots \\
q_{3,m3} \\
\vdots \\
q_{s,m4} \\
\vdots
\end{bmatrix} = \begin{bmatrix}
\vdots \\
\partial J/\partial p_{1,m1} \\
\vdots \\
\partial J/\partial p_{2,m2} \\
\vdots \\
\partial J/\partial p_{3,m3} \\
\vdots \\
\partial J/\partial p_{s,m4} \\
\vdots
\end{bmatrix}
\tag{97}
$$

Notice that because of the form of $M^{(2)}$ one can compute $q_{1,m1}$; $q_{2,m2}\cdots,q_{3,m3}\cdots$ independently. After these quantities have been found, one determines $..q_{s,m4}...$

For fixed elements in the x,y,z space the elements of the matrix $M^{(11)}$ are independent of the point in the $\vec{w}$ space for which tne gradient is determined. The matrices usually have block tri-diagonal shape. The left-right decomposition can be

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done once and for all. The elements adjacent to the shock change from iteration to iteration. The left-right decomposition for the matrix $M^{(11)}$ amounts to a Gaussian elimination of the unknowns (combined in column vectors) starting at the left or at the right. If one arranges the elimination sequence so that one eliminates last the column vectors affected by the changing shock elements, then most of the preparatory work for the solution of the system of equations will remain unaffected by a change of the shock position. An alternative procedure which leaves the block tri-diagonal form of the problem intact is shown in Appendix A.

The evaluation of the matrix $M^{(2)}$ can be avoided for the present choice of the scalar product if one defines the scalar product in the vector space of the p_j and $p_{s,j}$ rather than in the f nction space of $\vec{w}$ and f_s.

If $u = \sum\limits_{j1} p^{(1)}_{j1} \phi_{j1}$ and $v = \sum\limits_{j2} p^{(2)}_{j2} \phi_{j2}$

$$\qquad\qquad\qquad\qquad (98)$$

Then

$$[u,v] = \sum\limits_{j} p^{(1)}_{j} p^{(2)}_{j}$$

For reasonably chosen shape functions, the definitions are equivalent. For example, they guarantee identity of the two vectors under the same circumstances and they show the same convergence behavior in the summation over j. Then one finds for the components of g simply

$$q_{i,j} = \partial J / \partial p_{i,j}$$

$$\qquad\qquad\qquad\qquad (99)$$

$$q_{s,j} = \partial J / \partial p_{s,j}$$

It has been mentioned above that the definition Eq. (53) for the scalar product fails to correct for the overemphasis of short wave errors inherent in the definition of J and that this will probably have a detrimental effect on the search. Whether

or not this can be remedied by a multi-grid approach is best
determined by numerical experiment.

If one uses the distance definition of Eq. (57) (for the
elements $\vec{w}$, f_s), then the procedure is completely analogous.
Eq. (88) is replaced by

$$\sum_i [g_i, \delta w_i] + [g_s, \delta f_s] = \sum_i \int_\Omega (\text{grad } g_i \cdot \text{grad } \delta w_i) d\tau \qquad (100)$$

$$+ \int_\Gamma c_5 g_i \delta w_i d_o + \begin{cases} \int_{\Gamma_s} c_6 g_s \delta f_s dydz \\[2ex] \text{or } \int_{\Gamma_s} c_6 \text{grad } g_s \cdot \text{grad } \delta f_s dydz \end{cases}$$

Then

$$M^{(11)}_{j2,j1} = M^{(11)}_{j1,j2} = \int_\Omega \text{grad } \phi_{j1} \cdot \text{grad } \phi_{j2} d\tau + \int_\Gamma c_5 \phi_{j1} \phi_{j2} d_o$$

$$M^{(i,21)}_{k,j1} = \int_\Omega \text{grad} \phi_{j1} \cdot \sum_{j2} P_{i,j2} (\partial \text{grad } \phi_{j2} / \partial P_{s,k}) d\tau$$

$$M^{(2,2)}_{j1,j2} = \begin{cases} \int_{\Gamma_s} c_6 \psi_{j1} \psi_{j2} dydz \\[2ex] \text{or } \int_{\Gamma_s} c_6 \text{grad } \psi_{j1} \cdot \text{grad } \psi_{j2} dydz \end{cases}$$

In computing J (and $\partial J/\partial p$), one must, of course, include
the expressions Eq. (52) which are due to the presence of shocks.
If one includes in the formulation of the functional an approxi-
mation to the boundary data, then one evaluates the expression
for J (Eq. (31)), otherwise one evaluates Eq. (30). Depending on
the form which one uses, the shape parameters by which the
boundary data are expressed will be kept fixed or will be allowed
to vary. This will express itself also in the expression for the
g_i's and for g_s.

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The determination of the $g_{i,j}$ for a fixed i amounts to a solution of a Poisson equation. Two alternatives for $[g_s, \delta w_s]$ appear in Eq. (100). Since the expression δJ contains derivatives of f_s the second one (including grad f_s in the y,z space) is preferable from a theoretical point of view. The number of parameters describing the shape of the shock is not very large, therefore, there may not be a need for counteracting the waviness of the shock by the second choice of the scalar product $[g_{s,j} \delta w_s]$.

In computing J for flow fields containing shocks one must, of course, include the integrals over the shock surface (Eq. (52)), most likely in the modified form Eq. (73)). We have written down two forms for J; in (Eq. (30)) the function $\vec{w}$ satisfies the boundary conditions. The other one (Eq. (31)) includes terms which take residuals in the boundary conditions into account. In the first case some of the shape parameters p are not subject to variation, in the second case all shape parameters will be varied. The constants c_1 and c_2 in Eq. (31) express the weight which one gives to the residuals in the boundary conditions compared to the weight of the residuals in the flow field. Taking c_1 and c_2 too large probably will slow down the convergence of the procedure. The same holds for the functions c_3 and c_4.

It may be necessary to add further terms to J in order to stabilize the procedure in the supersonic region and/or to bias the procedure against expansion shocks. This will be discussed later.

According to the general theory, the determination of the gradient function g requires the solution of the Poisson equation (Eq. (59)). In the discretized form this means that one solves Eq. (97) for the parameters $q_{i,ml}$ which belong to the gradient function g_i. The governing matrices $M^{(11)}$ contain all the necessary information including the boundary conditions. The matrices are sparse (frequently of block tri-diagonal form). Whether or not one can apply super-fast Poisson solvers depends

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upon the relation between the matrices found in one row (they must commute). This is a severe limitation on the choice of the shape functions. (The author has not investigated this aspect.) Unless one tries to take advantage of this special form, one might introduce weight functions into the definition Eq. (57) of the scalar product. This will not be detrimental to the sparsity of the matrix, although it might complicate the computation of the matrix elements.

Two expressions have been given in Eq. (100) for the distance $[g_s, \delta f_s]$. The second is preferable from a theoretical point of view because the expression Eq. (75) contains derivatives of the function f_s. It, therefore, overemphasizes waves in the shock shape. In practice it is perhaps not necessary to counteract this phenomenon by the choice of the scalar product.

The gradient functions will be used in a conjugate gradient search for the minimum of J. Important are, of course, the values of the shape parameters p and p_s (not the actual value of the minimum of J). At the first stages only a moderate accuracy is needed because the method is self-correcting and one is likely to have frequent restarts in any case. The conjugate gradient search methods determine the center of hypersurfaces in a space of a great number of dimensions which have a shape similiar to very elongated ellipsoids. The scalar product introduced compensates for the difference in the lengths of the axes. All axes which have the same length are treated in the same manner. (This is the reason why one finds the minimum in one step if the surfaces of constant J is transformed into a hypersphere.) The ratio of the largest to the smallest axis of such hyper-ellipsoids does not change too much if one increases the number of dimensions for which the flow is computed. This makes it possible to test the effectiveness of the search procedure in relatively simple examples.

In problems governed by the Laplace equation, the Bristeau metric changes the hyperellipsoids obtained in the $\vec{w}$ space for

surfaces J = const with an unsophisticated norm into hyper-
spheres, with the consequence that the solution (i.e., the center
of the hypersphere) is found in one search step. In the general
case one can obtain the Laplace equation in the distant field, by
means of the Prandtl-Glauert approximation. This is the basis
for the application of far field conditions. It can be expected
that this transformation has a desirable effect on most of the
flow pattern. We study its effect on the present approach.

Let

$$\beta = (1 - M_0{}^2)^{1/2} \tag{101}$$

where M_0 is the free stream Mach number. (The reader will notice
that the letter M, but with superscript, has been used before to
define a matrix which arises if one linearizes the mass flow
vector $\vec{m}$. The author hopes that this will not cause confusion.)
As before, we define the local vector in the original flow field
by its Cartesian coordinates x,y,z. The coordinates in the
distorted system are denoted by $\tilde{x},\tilde{y},\tilde{z}$. The Prandtl-Glauert
distortion is then given by

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = D \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{102}$$

where D is an operator defined by the diagonal matrix.

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix} \tag{103}$$

Linearizing the basic flow differential Eq. (25) one obtains

$$\operatorname{div} M^{(1)} \vec{w} = 0 \tag{104}$$

where M, originally defined in Eq. (33), is specialized to

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$$M^{(1)} = \rho_0 \begin{pmatrix} 1-M_0^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \rho_0 \ \beta^2 D^{-2} \tag{105}$$

Here ρ_0 denotes the dimensionless free stream density. All quantities expressed in the distorted coordinates are characterized by a tilde. For instance

$$\tilde{\vec{w}}(\tilde{x},\tilde{y},\tilde{z}) \equiv \tilde{\vec{w}}\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \vec{w}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{w}\left(D^{-1}\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix}\right)$$

We denote by $\widetilde{\text{grad}}$, $\widetilde{\text{div}}$, $\widetilde{\text{curl}}$, the operations carried out in the distorted system. One has the following relations

$$\text{grad } \phi = D \ \widetilde{\text{grad}} \ \tilde{\phi}$$

$$\text{div } \vec{u} = \widetilde{\text{div}} \ D \ \tilde{\vec{u}} \tag{106}$$

$$\text{curl } \vec{u} = \beta^2 \ D^{-1} \ \widetilde{\text{curl}}(D^{-1}\tilde{\vec{u}})$$

Taking the symbols in the generic sense, one has the inverse relations

$$\widetilde{\text{grad}} \ \tilde{\phi} = D^{-1} \ \text{grad } \phi$$

$$\widetilde{\text{div}} \ \tilde{\vec{u}} = \text{div } D^{-1} \ \vec{u} \tag{107}$$

$$\widetilde{\text{curl}} \ \tilde{\vec{u}} = \beta^{-2} \ D \ \text{curl}(D \ \vec{u})$$

One has for instance, with Eq. (102) from

$$\phi(x,y,z) = \tilde{\phi}(\tilde{x},\tilde{y},\tilde{z})$$

$$\text{grad } \phi = \vec{e}_x \phi_x + \vec{e}_y \phi_y + \vec{e}_z \phi_z$$

$$= \vec{e}_x \tilde{\phi}_{\tilde{x}} + \vec{e}_y \beta \tilde{\phi}_{\tilde{y}} + \vec{e}_z \beta \tilde{\phi}_{\tilde{z}}$$

$$= D \ \widetilde{\text{grad}} \ \tilde{\phi}.$$

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Here the unit vectors in the x, y, and z directions are denoted, respectively, by $\vec{e}_x$, $\vec{e}_y$, and $\vec{e}_z$.

Let $h(x,y,z) = $ const define some surface in the x,y,z space. Because of Eq. (102) the corresponding surface in the $\tilde{x}, \tilde{y}, \tilde{z}$ representation is given by

$$h\begin{pmatrix}\tilde{x}\\\tilde{y}\\\tilde{z}\end{pmatrix} = h\left(D^{-1}\begin{pmatrix}\tilde{x}\\\tilde{y}\\\tilde{z}\end{pmatrix}\right)$$

One has for the normal vectors in the respective spaces

$$\vec{e}_n = \text{grad } h/|\text{grad } h|$$

$$\tilde{\vec{e}}_n = \text{gr}\tilde{a}\text{d } \tilde{h}/|\text{gr}\tilde{a}\text{d } \tilde{h}|$$

Then

$$\tilde{\vec{e}}_n = D^{-1} \text{ grad } h/|D^{-1} \text{ grad } h|$$

$$\tilde{\vec{e}}_n = D^{-1} \vec{e}_n(|\text{grad } h|/|D^{-1} \text{ grad } h|) \qquad (108)$$

Consider the integral J_1 (Eq. (30))

$$J_1 = \frac{1}{2}\int_\Omega [(\text{div } \vec{m})^2 + (\text{curl } \vec{w})^2]d\tau \qquad (109)$$

One has, with Eqs. (107)

$$\text{div } \vec{m} = \text{d}\tilde{\text{i}}\text{v}(D \, \tilde{\vec{m}})$$

$$\text{curl } \vec{w} = \beta^2 D^{-1} \text{ c}\tilde{\text{u}}\text{rl}(D^{-1} \, \tilde{\vec{w}}) \qquad (110)$$

In the linearized flow one has, from Eq. (105)

$$\vec{m} = \rho_0\beta^2 D^{-2}\vec{w} \qquad (111)$$

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Then, with Eq. (106)

$$\text{div } \vec{m} = \rho_0 \beta^2 \widetilde{\text{div}} \; \vec{\tilde{u}} \qquad (112)$$

with

$$\vec{\tilde{u}} = D^{-1} \vec{\tilde{w}} \qquad (113)$$

Furthermore,

$$\text{curl } \vec{w} = \beta^2 D^{-1} \text{ curl } \tilde{u}$$

One has

$$d\tau = dxdydz = \beta^{-2} d\check{x} d\check{y} d\check{z} = \beta^{-2} d\tilde{\tau}$$

Then

$$J_1 = (\beta^2/2) \int_{\Omega} [\rho_0^2 (\widetilde{\text{div}} \; \tilde{u})^2 + (D^{-1} \text{ curl } \tilde{u})^2] d\tilde{\tau}$$

Here different components of curl $\tilde{u}$ appear with different weights. Greater symmetry between the distorted coordinate directions is obtained if one defines

$$\bar{J}_1 = (\beta^2/2) \int_{\Omega} [\widetilde{\text{div}} \; \tilde{u})^2 + (\widetilde{\text{curl}} \; \tilde{u})^2] d\tilde{\tau} \qquad (114)$$

One then obtains, in terms of the original coordinates

$$\bar{J}_1 = (1/2) \int_{\Omega} [\rho_0^{-2} (\text{div } \vec{m}(\vec{w}))^2 + (D \text{ curl } \vec{w})^2] d\tau \qquad (115)$$

The original Eqs. (25) and (26) are left unchanged but the weights assigned in the definition of $\bar{J}_1$ to different components are now different. This is the definition for $\bar{J}_1$ (and correspondingly $\bar{J}_2$) which will be used in the future. The

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necessary modifications of the formulae of this section are left
to the reader. A corresponding modification is carried out in
the Bristeau distance definition Eq. (57). It is now natural to
define

$$[\tilde{\vec{u}}, \tilde{\vec{v}}] = \sum_{i=1}^{3} (\int_{\Omega} \text{grad } \tilde{\vec{u}}_i \cdot \text{grad } \tilde{v}_i) d\tilde{\tau} + \int_{\Gamma} c_5 \tilde{u}_i \tilde{v}_i d\tilde{o})$$

Returning to the original coordinates, using Eq. (107) one
obtains

$$[\vec{u}, \vec{v}]^{(1)} = \tag{116}$$

$$\sum_{i=1}^{3} (\int_{\Omega} ((D^{-1} \text{ grad } u_i) \cdot (D^{-1} \text{ grad } v_i) d\tau + \int_{\Gamma} c_5 u_i v_i do)$$

Actually $d\tilde{o} \neq do$, but the difference between these expressions
can be incorporated in the function c_5. Furthermore, a factor β^2
which arises because $d\tilde{\tau} = \beta^2 d\tau$ has been omitted. Accordingly,
one defines

$$[\vec{g}, \delta\vec{w}] = \tag{117}$$

$$\sum_{i=1}^{3} (\int_{\Omega} (D^{-1} \text{ grad } \vec{g}_i \cdot D^{-1} \text{ grad } \delta\vec{w}) d\tau + \int_{\Gamma} c_5 g_i \delta w_i do)$$

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FORMULATION OF THE PROBLEM IN TERMS OF THE VELOCITY
POTENTIAL ϕ AND AN AUXILIARY VECTOR POTENTIAL $\vec{A}$

In terms of the velocity potential ϕ the problem is given by

$$\text{div } \vec{m}(\text{grad } \phi) = 0 \qquad\qquad (118)$$

where $\vec{m}$ is defined as before (beginning of Section III). One prescribes as boundary conditions: ϕ on Γ_1; $\vec{m} \cdot \vec{e}_n$ on Γ_2. Eq. (118) is satisfied if one introduces a (single-valued) vector field $\vec{A}$ and postulates

$$\vec{m}(\text{grad } \phi) - \text{curl } \vec{A} = 0 \qquad\qquad (119)$$

with ϕ prescribed on Γ_1; curl $\vec{A} \cdot \vec{e}_n$ prescribed on Γ_2. More specifically, one can introduce in Γ_1 and Γ_2 two families of coordinate curves p_1 and p_2 and then write

$$\phi = f_1(p_1, p_2) \quad \text{on } \Gamma_1 \qquad\qquad (120)$$

$$\text{curl } \vec{A} \cdot \vec{e}_n = f_2(p_1, p_2) \quad \text{on } \Gamma_2 \qquad\qquad (121)$$

where $f_1(p_1, p_2)$ and $f_2(p_1, p_2)$ are given functions. ϕ and $\vec{A}$ are the independent variables used by Johnson and his co-workers.

The boundary condition, Eq. (121) is unusual because it requires a differentiation of the same order as that occurring in the partial differential equation (Eq. (119)). Moreover $\vec{A}$ occurs in Eq. (119) and (121) only in the form curl $\vec{A}$; it therefore is not uniquely determined. The following discussions show how these flaws in the formulation can be remedied.

We call a function $\vec{A}$ admissible, if it satisfies Eq. (121). Let $\vec{A}_0$ and $\vec{A}_1$ be two admissible functions. Consider a closed curve in Γ_2 whose interior lies entirely in Γ_2. Let S be the

surface cut out by this curve and $\vec{ds}$ the line element along this curve. Then by Stokes' theorem

$$\oint (\vec{A}_1 - \vec{A}_0) \cdot \vec{ds} = \int_S \text{curl}(\vec{A}_1 - \vec{A}_0) \cdot \vec{e}_n \, do$$

The right-hand side vanishes, because $\vec{A}_0$ and $\vec{A}_1$ are admissible functions. It follows that

$$\int_{x_0,y_0,z_0}^{x,y,z} (\vec{A}_1 - \vec{A}_0) \cdot \vec{ds} \qquad \begin{array}{l} x,y,z \subset \Gamma_2 \\ x_0,y_0,z_0 \subset \Gamma_2 \end{array}$$

is independent of the path between the two points. This defines a function $F(x,y,z)$ on Γ_2

$$F(x,y,z) - F(x_0,y_0,z_0) = \int_{x_0,y_0,z_0}^{x,y,z} (\vec{A}_1 - \vec{A}_0) \cdot \vec{ds}; \quad \begin{array}{l} x_0,y_0,z_0 \subset \Gamma_2 \\ x,y,z \subset \Gamma_2 \end{array}$$

(122)

The point (x_0,y_0,z_0) is fixed, $F(x_0,y_0,z_0)$ is a constant. The function F is single-valued only for a simply connected surface Γ_2 (Fig. 1a). An example of a surface Γ_2 which is not simply connected is shown in Figs. 1b and 1c. (A curve extending from a point of the boundary curve C_1 to a point of the boundary curve C_2 does not cut Γ_2 into two separate parts.) In Fig. 1c there are curves beginning and ending at a point P of Γ_2 which do not cut out a portion of the entire surface Γ that belongs solely to Γ_2. In the example such a portion of Γ will either contain the exit cross section or the entrance cross section. Denoting by C_3 the curve within Γ_2 which starts and ends at point P and by C_2 the boundary between Γ_2 and the exit cross section, one has

$$\int_{C_3} (\vec{A}_1 - \vec{A}_0) \cdot d\vec{s} - \int_{C_2} (\vec{A}_1 - \vec{A}_0) \cdot d\vec{s} = 0$$

or (123)

$$\int_{C_3} (\vec{A}_1 - \vec{A}_0) \cdot d\vec{s} = \text{const}$$

where the constant depends upon $\vec{A}_1$ and $\vec{A}_0$, but not on the location of the point P.

Let the surface Γ_2 be parametrized by two parameters p_1 and p_2 where p_2 changes by a constant L if one returns to the same point along any path which encloses the exit cross section. A developable surface Γ_2 can be spread out (without distortion) into a plane, then the coordinates can be easily visualized. (See for instance Fig. 2 for a cone. Actually, such a visualization is not needed.) If one starts at a point (p_1, p_2) and follows a path of the kind described above which ends at the same point of Γ_2, one then has coordinates $(p_1, p_2 + L.)$

According to Eq. (123), all functions F evaluated along such a path will increase by the same amount. Let F and $\tilde{F}$ be two functions (not necessarily computed in this manner) for which

$$F(p_1, p_2 + L) = F(p_1, p_2) + \text{const}$$

$$\tilde{F}(p_1, p_2 + L) = \tilde{F}(p_1, p_2) + \text{const}$$

where the constant is the same. Then trivially

$$F(p_1, p_2 + L) - \tilde{F}(p_1, p_2 + L) = F(p_1, p_2) - \tilde{F}(p_1, p_2)$$

in other words

$$F_2 = F(p_1, p_2) - \tilde{F}(p_1, p_2)$$

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is single-valued on Γ_2. It follows that each $F(x,y,z)$ can be represented in the form

$$F = \text{const } F_1(x,y,z) + F_2(x,y,z) \qquad (124)$$

where F_1 is some standard function for which

$$F_1(p_1,p_2 + L) = F_1(p_1,p_2) + L$$

$$(\text{for instance } F_1 = p_2)$$

and F_2 is an arbitrary (sufficiently smooth function) which is single-valued on Γ_2. The function F_2 can be continued through Ω. Now we define

$$\vec{A}_2 = \vec{A}_1 - \text{grad } F_2 \qquad (125)$$

One observes that

$$\text{curl } \vec{A}_2 = \text{curl } \vec{A}_1$$

Accordingly, the contributions curl $\vec{A}$ occurring in the original differential equation (Eq. (119)) and the boundary condition Eq. (121) are the same. Now consider

$$\int_{x_0,y_0,z_0}^{x,y,z} (\vec{A}_2 - \vec{A}_0) \cdot d\vec{s} = \int_{x_0,y_0,z_0}^{x,y,z} (\vec{A}_1 - \text{grad } F_2 - \vec{A}_0) \cdot d\vec{s}$$

Hence with Eqs. (122) and (124)

$$\int_{x_0,y_0,z_0}^{x,y,z} (\vec{A}_2 - \vec{A}_0) \cdot d\vec{s} =$$

$$F(x,y,z) - F(x_0,y_0,z_0) - (F_2(x,y,z) - F_2(x_0,y_0,z_0))$$

$$= \text{const}[F_1(x,y,z) - F_1(x_0,y_0,z_0)]$$

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or

$$\int_{x_0, y_0, z_0}^{x, y, z} (\vec{A}_2 - \vec{A}_0 - \text{const grad } F_1) \cdot d\vec{s} = 0 \qquad (126)$$

It does not matter that in Eq. (126) that F_1 is defined only on Γ_2, because the equation contains only the components of grad F_1 which lie in Γ_2. These components are single-valued (although F_1 is not). In order for Eq. (126) to be satisfied for every path one must have

$$\vec{A}_2 \times \vec{e}_n = \vec{A}_0 \times \vec{e}_n - \text{const grad } F_1 \times \vec{e}_n \qquad (127)$$

We have thus constructed from an arbitrary admissible function $\vec{A}_1$ another admissible function for which

$$\text{curl } \vec{A}_2 = \text{curl } \vec{A}_1$$

and whose components in the tangential plane to Γ_2 agree with those of an admissible function $\vec{A}_0$, except for the gradient of a standard function F_1 multiplied by an arbitrary constant. The fact that curl $\vec{A}_2$ = curl $\vec{A}_1$ guarantees that $\vec{A}_2$ gives the same contribution to the partial differential equation (Eq. (119)) and the boundary condition (Eq. (121)). Admissible functions $\vec{A}_1$ can be replaced by $\vec{A}_2$ without loss of generality. Eq. (127) is a boundary condition for $\vec{A}_2$, which relates its components in the tangential plane to one admissible function $\vec{A}_0$ except for a one-parameter standard contribution const(grad $F_1 \times \vec{e}_n$).

One will ask how a function $\vec{A}_0$ can be constructed. If Γ_2 is a plane (or a developable surface), one may choose for the P_1, P_2 system a Cartesian system of coordinates, (x_1, x_2), with the corresponding components of $\vec{A}_0$ given by $A_{0,1}$ and $A_{0,2}$. Then

$$-\partial A_{0,1}/\partial x_2 + \partial A_{0,2}/\partial x_1 = f(x_1, x_2) \qquad (128)$$

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One can then choose one of the components, for instance $A_{0,1} \equiv 0$ and determine $A_{0,2}$ by integrations. Of course if $f_2(p_1,p_2) \equiv 0$, then one can simply set the components of $\vec{A}_0$ within the surface Γ_2 equal to zero. Functions $\vec{A}_0$ can be obtained by integrations also for a more general surface, but the equations are more complicated.

The function $\vec{A}_2$ is not uniquely determined, even if one keeps $\vec{A}_1$, $\vec{A}_0$, the function F_1, and the constant in Eq. (126) fixed, because it depends upon the function F_2 (including its continuation through Ω). To eliminate this ambiguity we add the conditions

$$\text{div}(\beta^2 D^{-2} \vec{A}_2) \equiv 0 \qquad \text{in } \Omega \qquad (129)$$

and

$$D^{-1}\vec{A}_2 \cdot \vec{e}_n = 0 \qquad \text{on } \Gamma - \Gamma_2 \qquad (130)$$

In the last equations β^2 and D have been included in anticipation of an application of the Prandtl-Glauert coordinate distortion. One then obtains from Eq. (125)

$$\text{div}(\beta^2 D^{-2}\vec{A}_1) - \text{div}(\beta^2 D^{-2} \text{ grad } F_2) = 0$$

or

$$(1-M_0^2)F_{2,xx} + F_{2,yy} + F_{2,zz} = \text{div}(\beta^2 D^{-2}\vec{A}_1) \quad \text{in } \Omega$$

and

$$\text{grad } F_2 \cdot (D^{-1}\vec{e}_n) = \vec{A}_1 \cdot D^{-1}\vec{e}_n \qquad \text{on } \Gamma_2$$

We demonstrate that F_2 and with it $\vec{A}_2$ are uniquely determined, by the additional conditions (Eqs. (129) and (130)). On Γ_2 the function F_2 can be determined from Eq. (124) as follows. For given $\vec{A}_1$ and $\vec{A}_0$ the function F on the left of Eq. (124) is determined from Eq. (122); it will increase by some

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constant if p_2 is replaced by $p_2 + L$. The constant in Eq. (124) is determined from the equation

$$F(p_1, p_2 + L) - F(p_1, p_2) = \text{const}(F_1(p_1, p_2 + L) - F_1(p_1, p_2))$$

(One remembers that F_1 is a prechosen standard function.) One thus obtains F_2 on Γ_2. It follows that F_2 is determined by a well-posed boundary value problem for a Poisson equation.

The original problem can now be reformulated:

$$\vec{m}(\text{grad } \phi) - \text{curl } \vec{A} = 0 \tag{131}$$

$$\text{div } \beta^2 D^{-2} \vec{A} = 0 \tag{132}$$

$$\vec{A} \times \vec{e}_n = \vec{A}_0 \times \vec{e}_n + \text{const grad } F_1 \times \vec{e}_n \quad \text{on } \Gamma_2 \tag{133}$$

$$D^{-1} \vec{A} \cdot \vec{e}_n = 0 \quad \text{on } \Gamma - \Gamma_2 \tag{134}$$

$$\phi = f_1 \quad \text{on } \Gamma_1 \tag{135}$$

Eqs. (131) and (132) give four equations for the three components of $\vec{A}$ and for ϕ. These equations represent the formulation of the original problem. The idea of residual minimization has not yet been introduced.

The introduction of the additional conditions Eqs. (133) and (134) which make $\vec{A}$ unique is desirable even from a computational point of view. If $\vec{A}$ is not uniquely determined, then the system which determines $\vec{A}$ is degenerate, viz., the equations are linearly dependent. Because of truncation and/or rounding errors, the system will in practice not be degenerate although its determinant will be close to zero. In this case one obtains only one solution for $\vec{A}$ and it is close to some solution of the ideal system. But the approximation for $\vec{A}$ so obtained depends upon the truncation and/or rounding errors; it may change wildly from one iteration step to the next (although the important quantity curl $\vec{A}$ may change only little). Such fluctuations of

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Shock conditions are expressed simply by the requirement that ϕ and the components of $\vec{A}$ in the shock surface are continuous at the shock. The first condition guarantees that the components of the velocity tangential to the shock are continuous, the second one that curl $\vec{A} \cdot \vec{e}_{s,n}$ remains unchanged. In analogy to Eq. (132) one can even postulate that there is no jump of the component of $D^{-2}\vec{A}$ normal to the shock. (This amounts to the requirement that Eq. (134) be satisfied as one passes through the shock.) It follows that in this formulation the shock need not be excised from the flow field. Of course one needs rather small elements in order to obtain a good definition of the shock shape.

To determine the form of the functional J which takes the Prandtl-Glauert coordinate distortions into account, we go back to Eq. (131). In dealing with a flow field which approaches a parallel flow as one goes to infinity, it is practical to deal with only the deviation from this parallel flow, although these deviations need not be small. Accordingly, we set

$$\phi = xw_\infty \cdot \vec{e}_x + \phi' \tag{136}$$

$$\vec{m}(w_0 \cdot \vec{e}_x + \text{grad } \phi') = \rho_0 w_0 \vec{e}_x + \vec{m}'(\text{grad } \phi') \tag{137}$$

and

$$\vec{A} = \frac{1}{2}(y\vec{e}_z - z\vec{e}_y)\rho_0 w_0 + \vec{A}' \tag{138}$$

One then has

$$\text{div } \vec{m} = \text{div } \vec{m}'(\text{grad } \phi') = 0 \tag{139}$$

and in analogy to Eq. (131)

$$\vec{m}'(\text{grad } \phi') - \text{curl } \vec{A}' = 0. \tag{140}$$

For a linearized parallel flow one obtains (with Eqs. (104) and (105))

For a linearized parallel flow one obtains (with Eqs. (104) and (105))

$$\vec{m}' = \rho_0 \beta^2 D^{-2}(\text{grad } \phi') \tag{141}$$

One has with Eqs. (106)

$$\tilde{\vec{m}}' = \rho_0 \beta^2 D^{-1}(\text{grad } \phi') \tag{141a}$$

Hence from Eq. (141), taking into account that ρ_0 and β are constant

$$\text{div } \vec{m}' = \text{div}(D^{-2} \text{ grad } \phi')$$

and with Eqs. (106)

$$\widetilde{\text{div}}(\widetilde{\text{grad}} \, \tilde{\phi}') = 0$$

We set in analogy with Eq. (131)

$$\widetilde{\text{grad}} \, \tilde{\phi}' - \widetilde{\text{curl}} \, \tilde{B}' = 0$$

This suggests that one define

$$\tilde{J} = \int_\Omega [(\widetilde{\text{grad}} \, \tilde{\phi}' - \widetilde{\text{curl}} \, \tilde{B})^2 + (\text{div } \tilde{B})^2] d\tau'$$

The second term in the integrand is included to make $\tilde{B}$ unique. Returning to $\vec{m}'$, first in the linearized form, but then for the general problem, one obtains, with Eq. (141a)

$$\tilde{J} = \int_\Omega [(\rho_0^{-1} \beta^{-2} D \tilde{\vec{m}}' - \widetilde{\text{curl}} \, \tilde{B})^2 + (\text{div } \tilde{\vec{B}})^2] d\tilde{\tau}$$

We now return to the original coordinates, using Eqs. (107)

$$\bar{J} = [(\rho_0^{-1} \beta^{-2} D \vec{m}' - \beta^{-2} D \text{ curl } DB)^{-2} + \text{div}(D^{-1} \vec{B})^2] \beta^2 d\tau$$

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We set

$$\rho_0 D\vec{B} = \vec{A}'$$

After multiplying with a constant $\rho_0{}^2\beta^2$, which is unessential in the minimization process, one obtains

$$J = \frac{1}{2} \int_\Omega \{[D(\vec{m}' - \text{curl } \vec{A}')]^2 + (\text{div}(\beta^2 D^{-2}\vec{A}'))^2\}d\tau \qquad (142)$$

In this formulation the expression Eq. (140) appears, except that it is modified by the operator D, which gives different weignts to different components. The second term, introduced to make $\vec{B}$ and consequently also $\vec{A}$ unique, appears in the form anticipated in Eq. (132).

The Euler equations of the variational problem can be interpreted as follows. We can disregard Eqs. (132) and (134) which are introduced to define $\vec{A}$ uniquely, but are not needed to define the flow field. In the numerical work these conditions ought to be introduced explicitly. Then one has to minimize

$$\int [D(\vec{m}(\text{grad } \phi) - \text{curl } \vec{A}]^2 d\tau \qquad (143)$$

To be varied are ϕ and $\vec{A}$. Then

$$\delta J = \int_\Omega [D(\vec{m}(\text{grad } \phi) - \text{curl } \vec{A})]^2 \cdot [D(M \text{ grad } \delta\phi - \text{curl } \delta\vec{A}] d\tau$$

We use Eqs. (36) and (37), the fact that M and D are symmetric matrices, and the boundary conditions for $\delta\phi$ and $\delta\vec{A}$ resulting from Eqs. (133) and (135). It follows from Eq. (133) that

$$\delta\vec{A} \times \vec{e}_n = (\delta \text{ const}) (\text{grad } F_1 \times \vec{e}_n) \qquad \text{on } \Gamma_2$$

Then one obtains

$$\delta J = \int_{\Gamma - \Gamma_1} \delta\phi(MD^2(\vec{m} - \text{curl } \vec{A})\cdot\vec{e}_n)do - \int_\Omega \text{div}(D^2 M(\vec{m} - \text{curl } \vec{A}))\delta\phi$$

$$\text{(144)}$$

$$- \int_{\Gamma - \Gamma_2} [(D^2(\vec{m} - \text{curl } \vec{A})), \delta\vec{A}, \vec{e}_n]do - \int_\Omega \text{curl}(D^2(\vec{m} - \text{curl } \vec{A}))\cdot\delta\vec{A} = 0$$

This gives the Euler equations

$$\text{div}(MD^2(\vec{m} - \text{curl } \vec{A})) = 0$$

and

$$\text{(145)}$$

$$\text{curl}(D^2(\vec{m} - \text{curl } \vec{A})) = 0.$$

Let $\vec{r}$ be the modified residual in the original equations

$$\vec{r} = D^2(\vec{m} - \text{curl } \vec{A}) \qquad \text{(146)}$$

Then, from Eqs. (145)

$$\text{div}[M \vec{r}] = 0$$

$$\text{(147)}$$

$$\text{curl } \vec{r} = 0.$$

Denoting by $\vec{W}$ the deviation of $\vec{w}$ in a perturbed flow field from the values of $\vec{w}$ in an existing flow field, and linearizing Eqs. (25) and (26), one obtains

$$\text{div}[M \vec{W}] = 0$$

$$\text{(148)}$$

$$\text{curl } \vec{W} = 0.$$

One finds that the Euler Eqs. (147) and the linearized Eq. (148) are identical if one replaces $\vec{r}$ by $\vec{W}$. The boundary conditions for the variational problems must be of such a nature that they guarantee $\vec{r} \equiv \vec{0}$.

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One obtains from Eq. (144)

$$M \, \vec{r} \cdot \vec{e}_n = 0 \quad \text{on } \Gamma - \Gamma_1$$

$$\vec{r} \times \vec{e}_n = 0 \quad \text{on } \Gamma - \Gamma_2. \tag{149}$$

$$\int_{\Gamma_2} \vec{r} \cdot (\text{grad } F_1 \times \vec{e}_n) do = 0 \tag{150}$$

The last condition is obtained from the equation preceding Eq. (144). In a subsonic flow one has $\Gamma - \Gamma_2 = \Gamma_1$ and $\Gamma - \Gamma_1 = \Gamma_2$. The analogon to the boundary conditions Eq. (149) arises by linearizing the boundary conditions Eq. (28) and (29) for $\vec{w}$ (with the inhomogeneous terms set equal to zero. Accordingly, $\vec{r}$ satisfies the same kind of boundary value problem as the function $\vec{w}$. In the discussion following Eqs. (28) and (29) we had pointed out, that one additional condition is needed to give a uniquely determined solution, and given examples of such conditions. In the present context a condition of this kind arises again, namely Eq. (150). These conditions guarantee that $\vec{r} \equiv 0$.

The corresponding two-dimensional formulation is given by

$$\vec{m}(\text{grad } \phi) - (\psi_y \vec{e}_x - \vec{\psi}_x \vec{e}_y) = 0 \tag{151}$$

We consider the example of a flow through a duct (Fig. 3) and prescribe ϕ at the entrance and at the end sections. The walls are given by

$$x = x(s) \text{ and } y = y(s)$$

Then one has

$$\vec{m}(\text{grad } \phi) \cdot \vec{e}_n = f(s) \tag{152}$$

The parameter s gives the length along the walls. Then

$$\vec{e}_n = -(dy/ds)\vec{e}_x + (dx/ds)\vec{e}_y$$

and from Eqs. (127) and (128)

$$-\psi_y(dy/ds) - \psi_x(dx/ds) = f(s)$$

$$\psi = -F(s) + c \tag{153}$$

where

$$F(s) = \int f(s)ds \tag{154}$$

and c is arbitrary. Eq. (153) (formulated for the upper and lower wall) is the boundary condition in terms of ψ. Since the original system of differential equations contains only derivatives of ψ, one of the constants can be chosen zero. Minimizing the residual (without introducing the Prandtl-Glauert distortion) one now postulates

$$\int_\Omega (\vec{m}(\text{grad } \phi) - \vec{e}_x\psi_y + \vec{e}_y\psi_x))^2 dxdy = \text{Min}$$

Hence

$$\int_\Omega (\vec{m}(\text{grad } \phi - \vec{e}_x\psi_y + \vec{e}_y\psi_x) \cdot (M \text{ grad } \delta\phi - \vec{e}_x\delta\psi_y + \vec{e}_y\delta\psi_x)dxdy = 0$$

Let

$$\vec{r} = \vec{m}(\text{grad } \phi) - \vec{e}_x\psi_y + \vec{e}_y\psi_x$$

with components $\vec{r}_1$ and $\vec{r}_2$. Then by carrying out integrations by part

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$$\int_{\Gamma} (M \, \vec{r} \cdot \vec{e}_n) \, \delta\phi \, ds + \int_{\Gamma} (\vec{r}_1 \, dx + \vec{r}_2 \, dy) \, \delta\psi$$

$$- \int_{\Omega} (\operatorname{div} M \, \vec{r}) \, \delta\phi \, d\tau + \int_{\Omega} \left(\frac{\partial \vec{r}_1}{\partial y} - \frac{\partial \vec{r}_2}{\partial x} \right) \delta\psi \, d\tau = 0$$

Now

$$\delta\phi = 0 \quad \text{on } \Gamma_1$$

$$\delta\psi = \delta c \quad \text{on the upper boundary}$$

Hence one obtains the Euler equations

$$\operatorname{div}(M \, \vec{r}) = 0$$

$$\operatorname{curl} \vec{r} = 0$$

$$(M \, \vec{r}) \cdot \vec{e}_n = 0 \quad \text{on } \Gamma - \Gamma_1$$

$$\vec{r}_1 \frac{dx}{ds} + \vec{r}_2 \frac{dy}{ds} = 0 \quad \text{on } \Gamma - \Gamma_2$$

$$\int_{\Gamma_{2,1}} \left(\vec{r}_1 \frac{dx}{ds} + \vec{r}_2 \frac{dy}{ds} \right) = 0 \quad \text{on } \Gamma_{2,1}$$

where $\Gamma_{2,1}$ refers to the upper boundary. If one considers $\vec{r}$ as the analogue to $\vec{w}$, then one has exactly the linearized form of the differential equation and of the boundary conditions in the $\vec{w}$ formulation. The last condition expresses the analogue of the potential difference between entrance and exit cross section, evaluated by a path which follows the upper contour.

It should be possible to interpret the two-dimensional formulation as a special case of the three-dimensional problem. For this purpose we introduce two planes $z = 0$ and $z = z_0$ as additional boundaries and postulate along these planes $m \cdot e_n = 0$ (Fig. 4). The surface then consists of these planes and the upper and lower boundary of the channel. The function $\vec{A}$ now assumes the form

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$$\vec{A} = \psi(x,y)\vec{e}_z \tag{155}$$

In the flow field this obviously satisfies Eqs. (132) and, at the entrance and exit cross sections, Eq. (134). The function $\vec{A}_0$ which occurs in the boundary condition Eq. (133) must satisfy

$$\text{curl } \vec{A}_0 \cdot \vec{e}_n = \vec{m} \cdot \vec{e}_n$$

One has

$$\text{curl } \vec{A}_0 = \vec{e}_x(\frac{\partial}{\partial y}\vec{A}_{03} - \frac{\partial}{\partial z}\vec{A}_{02}) + \vec{e}_y(\frac{\partial}{\partial z}\vec{A}_{01} - \frac{\partial}{\partial x}\vec{A}_{03})$$
$$+ \vec{e}_z(\frac{\partial}{\partial x}\vec{A}_{02} - \frac{\partial}{\partial y}\vec{A}_{01})$$

(As always, the second subscript 1,2,3 refer respectively to the x,y,z components.) If one chooses

$$\vec{A}_{01} = 0, \quad \vec{A}_{02} = 0, \quad \vec{A}_{03} = -F(x,y) \tag{156}$$

at the upper and lower boundary, then

$$\text{curl } \vec{A}_0 = - \vec{e}_x \frac{\partial F}{\partial y} + \vec{e}_y \frac{\partial F}{\partial x} \quad .$$

Since

$$\vec{e}_n = - \vec{e}_x \frac{dy}{ds} - \vec{e}_y \frac{dx}{ds}$$

one obtains

$$\text{curl } \vec{A}_0 \cdot \vec{e}_n = - \frac{\partial F}{\partial s} = + f(s)$$

This together with Eq. (152) shows that for the upper and lower surface the choice Eq. (156) for $\vec{A}_0$ is correct. One sets $\vec{A}_0 \times \vec{e}_n = 0$ for the surfaces $z = 0$ and $z = z_0$.

The function $F_1(x,y,z)$ which also occurs in the boundary condition Eq. (133) has the following property. It must increase

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by a constant if one travels along a closed path around the walls; one may for instance travel in the positive z direction in the upper boundary, in the negative y direction in the boundary $z = z_0$, in the negative z direction in the lower boundary, and in the positive z direction in the plane z = 0. A suitable form for such a function is

$$F_1 = z \qquad \text{upper boundary}$$

$$F_1 = z_0 \qquad \text{plane } z_0 = \text{const}$$

$$F_1 = z_0 \qquad \text{lower boundary}$$

$$F_1 = z_0 \qquad \text{boundary } z = 0$$

Then grad $F_1 = \vec{e}_z$ at the upper boundary and zero in all other surfaces. Then from Eq. (133), for the components of $\vec{A}$ lying in the upper boundary surface

$$\vec{A} \cdot \vec{e}_z = -\vec{e}_z F + c\vec{e}_z$$

or

$$\psi = -F + c$$

as in Eq. (153). The application to the other surfaces Γ_2 is self-evident.

The entrance cross section for a two- or three-dimensional supersonic flow in a tunnel belongs to Γ_1 and Γ_2. (Both the potential and the mass flow are prescribed.) The surface Γ_2 (entrance cross section and side walls) is simply connected. The function F_1 and, consequently, the condition Eq. (150) do not appear. Nothing is prescribed in the exit cross section, because it belongs to $\Gamma - \Gamma_1$ as well as $\Gamma - \Gamma_2$. The boundary problem for $\vec{r}$ is that for a supersonic flow linearized in the vicinity of an

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existing flow, but the boundary conditions are prescribed as if the flow were reversed (Fig. 6).

Further discussions are needed in problems in which a transition from a supersonic to a subsonic flow occurs. In a supersonic flow we call surfaces, respectively, space-like or time-like, if the velocity component in the direction normal to it is larger or smaller than the velocity of sound. (All curves in a space-like surface lie outside of the fore and aft Mach cone pertaining to its points; in a time-like surface there are curves which lie within these Mach cones). The entrance and the exit cross sections of a supersonic channel are space-like; the side walls time-like. A shock surface is space-like with respect to the flow upstream of it, it is time-like with respect to the flow downstream (if this flow is supersonic).

In the presence of shocks, additional terms arise in the equation for δJ. The shock surface belongs neither to $\Gamma - \Gamma_1$ nor to $\Gamma - \Gamma_2$; $\delta \phi$ and $\delta \vec{A}$ are the same on the upstream and on the downstream side of the shock, but they can vary freely. Therefore, one obtains *additional* terms

$$\int_{\Gamma_s} \delta \phi \left([MD^2(\vec{m} - \text{curl } \vec{A}) \cdot \vec{e}_n]_-^+ \, do_s \right.$$

and

$$\int_{\Gamma_s} \delta \vec{A} \cdot [D^2(\vec{m} - \text{curl } \vec{A}) \times \vec{e}_n]_-^+ \, do_s$$

where the definitions of $[\ldots]_-^+$ and do_s are the same as in Section III. This gives further conditions

$$[MD^2(\vec{m} - \text{curl } \vec{A}) \cdot \vec{e}_n]_-^+ = 0 \qquad (157)$$

$$\text{on } \Gamma_s$$

$$[D^2(\vec{m} - \text{curl } \vec{A}) \times \vec{e}_n]_-^+ = 0 \qquad (158)$$

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or if one uses the definition Eq. (146)

$$[M \vec{r}]_-^+ \cdot \vec{e}_{ns} = 0 \tag{159}$$

$$[\vec{r}]_-^+ \times \vec{e}_n = 0. \tag{160}$$

The same equations are obtained by linearizing the shock conditions Eq. (51) (in the $\vec{w}$ formulation).

A further term δI arises in the presence of shocks because one must admit deformations of the shock, and one does not postulate that admissible functions satisfy at the shock (or anywhere else)

$$\vec{m}(\text{grad } \phi) - \text{curl } \vec{A} = 0.$$

This correction is given by

$$\int_{\Gamma_s} (D(\vec{m}(\text{grad } \phi) - \text{curl } \vec{A}))^2 \, \delta f_s dy dz$$

where f_s is defined in Eq. (49). This yields a further condition

$$.[D(\vec{m}(\text{grad } \phi) - \text{curl } \vec{A})^2]_-^+ = 0 \quad \text{on } \Gamma_2 \tag{161}$$

Again we study the boundary value problem for the residuals $\vec{r}$ given by the Euler equations. These equations will be satisfied when the minimum is attained. Consider the flow in a divergent channel which returns from supersonic to subsonic speeds through a shock (see Fig. 7). At the entrance cross section one prescribes ϕ and $\vec{A} \times \vec{e}_n$, at the side wall $\vec{A} \times \vec{e}_n$ and at the exit section ϕ. The values of ϕ at the exit cross section determines the location of the shock in the channel. The exit section belongs to $\Gamma - \Gamma_2$; there one must prescribe because of Eqs. (149) $\vec{r} \times \vec{e}_n = 0$; at the side walls one must prescribe $M \vec{r} \cdot \vec{e}_n = 0$ as this belongs to the portion of the contour $\Gamma - \Gamma_1$. No condition for $\vec{r}$ can be prescribed in the entrance

cross section for it belongs (as every supersonic entrance cross section) neither to $\Gamma - \Gamma_1$ nor to $\Gamma - \Gamma_2$.

The subsonic region has as boundary the exit section, part of the side walls, and the downstream side of the shock. The data prescribed so far are not sufficient to guarantee $\vec{r} \equiv 0$ in the subsonic region. To obtain a well-posed problem for $\vec{r}$ one must prescribe some condition at the remaining boundary of the subonic region, which is given by the downstream side of the shock, for instance

$$M \vec{r} \cdot \vec{e}_{ns} = 0 \quad \text{at } \Gamma_s^-. \tag{162}$$

Then one has $\vec{r} \equiv 0$ in the entire subsonic region, including, of course, at the downstream side of the shock. But $\vec{r}$ satisfies the linearized shock conditions Eqs. (159) and (160). Since the shock is a space-like surface with respect to the flow upstream of it, one obtains a well-posed boundary value problem for $\vec{r}$ (although with a reversed flow direction in the supersonic region). This then guarantees that $\vec{r} = 0$ throughout the flow field.

Actually one minimizes J not by solving the Euler equations but by a search procedure. Then the condition Eq. (162) does not appear, implicitly one satisfies by Eq. (161). This equation is certainly satisfied for $r^+ = r^- = 0$ together with the other boundary conditions for $\vec{r}$.

Let us now discuss numerical aspects, although so far no measures have been introduced to exclude expansion shock. In the numerical treatment one will impose the restrictions on $\vec{A}$ discussed above, namely

$$\text{div } D^{-2} \vec{A} = 0 \tag{163}$$

and

$$D^{-1} \vec{A} \cdot \vec{e}_n = 0 \quad \text{on } \Gamma - \Gamma_2 \tag{164}$$

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In addition one has, of course, the boundary conditions of the problem

$$\phi = f_1 \quad \text{on} \quad \Gamma_1 \tag{165}$$

$$\vec{A} \times \vec{e}_n = \vec{A}_0 \times \vec{e}_n + c \, \text{grad} \, F_1 \times \vec{e}_n \quad \text{on} \quad \Gamma_2 \tag{166}$$

with $\vec{f}_1$, $\vec{A}_0$, and F_1 known. The functional to be minimized is then

$$J = (1/2) \int_{\Omega} [D(\vec{m}(\text{grad} \, \phi) - \text{curl} \, \vec{A})^2 + (\text{div} \, B^2 D^{-2} \vec{A})^2] d\tau \tag{167}$$

If the boundary condition Eqs. (165) and (166) are of a nature that cannot be expressed exactly by means of the shape functions chosen in the discretization, then one will add integrals over the boundary which weight the residuals, arising in the boundary conditions, in the same manner as in Section III. An element of the function space is now given by the functions ϕ and $\vec{A}$ which satisfy the boundary conditions. Therefore, implicit in this definition are the functions f_1 (which occurs in the boundary condition for ϕ) and the function $\vec{A}_0 \times \vec{e}_n$ (which occur in the boundary condition for $\vec{A}$). If Γ_2 is not simply connected, the boundary values of $\vec{A}$ are furthermore determined by the function $\text{grad} \, F_1 \times \vec{e}_n$ and the constant c. The function $\text{grad} \, F_1 \times \vec{e}_n$ is fixed in a specific problem. The constant c is allowed to vary and, therefore, is included in the parameters used to characterize an element of the function space. If one chooses to consider the shock as an interior boundary, then the shock shape (the function $f_s(y,z)$) belongs to the characterization of an element of the function space, and the shape functions pertaining to elements adjacent to the shock will depend upon the characterization of f_s. The unknown functions are now expressed by shape functions x and shape parameters p.

$$\vec{A}_i(x,y,z) = \sum_j p_{i,j} X_j(x,y,z); \quad i = 1,2,3$$

<div align="right">(168)</div>

$$\phi(x,y,z) = \sum_j p_{4,j} X_j(x,y,z)$$

Some of the shape parameters $p_{i,j}$ are fixed by the boundary conditions. In cases where Γ_2 is not simply connected some parameters are expressed by the constant c. The definition for J in Eq. (167) presumes that one does not specify the shock location, in other words one carries out a shock-capturing procedure. If one prefers to consider the shock as an interior boundary, then one will have elements which have the shock surface as their boundaries, and some of the shape function will depend upon the parameters which describe the shape of the shock. Given a set of values $p_{i,j}$ (i = 1,...4,), and if one so chooses, the shape functions and the parameters which characterize the shock, one can evaluate J. The derivatives $\partial J/\partial p_{i,j}$ which are also needed in the search process are best found by numerical differentiation. The solution of the discretized problem is given by

$$\partial J/\partial p_{i,j} = 0$$

for all i's (including eventually the shock parameters) and all j's. If one specifies the shock location, then one has as shock conditions

$$[\phi]_-^+ = 0, \quad [\vec{A}]_-^+ = 0$$

The last equations will be directly taken into account when one computes J.

To define a gradient in this space, which is necessary if one carries out a conjugate gradient search, one must introduce a

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metric. A simple-minded definition of the scalar product between two elements $\vec{A}^{(1)}, \phi^{(1)}, f_s^{(1)}$ and $\vec{A}^{(2)}, \phi^{(2)}, f_s^{(2)}$ is given by

$$[(\vec{A}^{(1)}, \phi^{(1)}, f_s^{(1)}), (\vec{A}^{(2)}, \phi^{(2)}, f_s^{(2)})] \qquad (169)$$

$$= \int_{\Omega} \{\phi^{(1)}(x,y,z)\phi^{(2)}(x,y,z) + \sum_i \vec{A}_i^{(1)}(x,y,z)\vec{A}_i^{(2)}(x,y,z)\}d\tau$$

$$+ \int_{\Gamma_s} c_s(y,z)f_s^{(1)}(y,z)f_s^{(2)}(y,z)dydz$$

An equivalent scalar product can be defined in the space of the $p_{i,j}$'s.

$$[(p_i^{(1)}, p_s^{(1)}), (p_i^{(2)}, p_s^{(2)})]$$

$$= \sum_i \sum_j p_{i,j}^{(1)} p_{i,j}^{(2)} + \text{const} \sum_j p_{s,j}^{(1)} p_{s,j}^{(2)} \qquad (170)$$

The gradient is, of course, characterized in the same manner as the elements of the function space. Corresponding to $\vec{A}$, ϕ, and f_s we characterize the gradient by $g_{\vec{A}}$, g_ϕ, g_{fs}. Denoting again the shape parameters for the g's by q with appropriate subscripts one has

$$g_{\vec{A},i}(x,y,z) = \sum_j q_{i,j} x_j(x,y,z) \qquad i = 1,2,3$$

$$g_\phi(x,y,z) = \sum_j q_{4,j} x_j(x,y,z)$$

$$g_{fs}(y,z) = \sum_j q_{s,j} x_{s,j}(y,z)$$

Immediately using the definition for the scalar product, in Eq. (170) we express a change of J, once by

$$\delta J = \sum_{ij} (\partial J/\partial p_{i,j}) \delta p_{i,j} + \sum_j \partial J/\partial p_{s,y} \,\delta p_{s,y}$$

a second time by

$$[(q_{i,j}, q_{s,j}), (\delta p_{i,j}, \delta p_{s,j})] = \sum_{i,j} q_{i,j} \, \delta p_{i,j} + \text{const} \sum_j q_{s,j} \, \delta p_{s,j}$$

one obtains

$$q_{i,j} = \partial J / \partial p_{i,j}$$

$$q_{s,j} = \text{const}^{-1} \, \partial J / \partial \rho_{s,j}$$

This is the gradient in the $p_{i,j}$ space. As in Section III, the determination of δJ in terms of $\delta \phi$, $\delta \vec{A}$, and δf_s requires the formation of second derivatives (see Eq. (144)). No such derivatives are encountered in the definition of the scalar product used here. For the gradient function obtained by the present procedure, certain derivatives which exist for ϕ and $\vec{A}$ will not exist. Theoretically this holds only for the infinite-dimensional function space; in the discretized version the short-comings of the definition of the scalar product manifest itself in the difficulties encountered if one tries to find the minimum in a space for which surfaces of constant J are similar to very elongated ellipsoids. This difficulty can perhaps be overcome by using multigrid methods. This means that in part of the computa-tions one uses a vector space with a lower number of dimensions, in which the shortest axes of the ellipsoids are not present.

A better founded approach is obtained by using a definition of the scalar product similar to that of Bristeau.

$$[(\vec{A}^{(1)}, \phi^{(1)}, f_s^{(1)}), (\vec{A}^{(2)}, \phi^{(2)}, f_s^{(2)})] \qquad (171)$$

$$= \int_\Omega ((\sum_i D^{-1} \text{grad } A_i^{(1)} \cdot D^{-1} \text{grad } A_i^{(2)}) + \text{grad } D^{-1} \phi^{(1)}$$

$$\cdot \text{grad } D^{-1} \phi^{(2)}) d\tau + \text{const} \int_{\Gamma_s} f_s^{(1)} f_s^{(2)} \, dy dz$$

This form takes a Prandtl-Glauert coordinate distortion into account. The expression δJ does not contain derivatives of f_s. No derivatives of f_s (as in grad f_s) are, therefore, needed in the definition of the scalar product. Then

$$[(\vec{g}_A, g_\phi, g_s), (\delta \vec{A}, \delta \phi, \delta f_s)] \tag{172}$$

$$= \int_\Omega (\sum_i (D^{-2} \text{grad } g_{Ai} \cdot \text{grad } \delta A_i) + (D^{-2} \text{grad } g_\phi) \cdot \text{grad } \delta \phi) d\tau$$

$$+ \int_\tau g_s \, \delta f_s dy dz$$

The integrals are again expressed in terms of shape parameters; one obtains a sparse system of equations for the q's. Details are omitted because of the close analogy to the procedure of Section III. The equations for $g_{\vec{A}}$ and g_ϕ are in theory Laplace equations (in the distorted system) and in practice approximations thereof.

SECTION V
THE APPROACH OF BRISTEAU

Bristeau (Reference 2) starts from the differential equation

$$\text{div}(\vec{m}(\text{grad } \phi)) = 0 \qquad (173)$$

with boundary conditions

$$\phi = f_1(x,y,z) \quad \text{on } \Gamma_1 \qquad (174)$$

$$\vec{m} \cdot \vec{e}_n = f_2(x,y,z) \quad \text{on } \Gamma_2 \qquad (175)$$

The portions Γ_1 and Γ_2 of the boundary Γ are chosen as in Sections III and IV, so that one obtains a well-posed problem.

The Bristeau procedure is immediately formulated as if it were carried out in the distorted coordinates. One has, according to Eq. (123)

$$\text{div } \vec{m} = \widetilde{\text{div}} \, D\vec{\widetilde{m}}.$$

We associate with the residual in Eq. (173) a function ξ by setting

$$\widetilde{\text{div}} \, \widetilde{\text{grad}} \, \widetilde{\xi} = \widetilde{\text{div}} \, D\vec{\widetilde{m}} \qquad (176)$$

or in the original coordinates

$$\text{div } D^{-2} \text{ grad } \xi = \text{div } \vec{m}(\text{grad } \phi) \qquad (177)$$

subject to certain boundary conditions, which guarantee that $\widetilde{\text{grad}} \, \xi = 0$ if $\widetilde{\text{div}} \, D\vec{\widetilde{m}} = 0$. They will be discussed later. One now minimizes

$$\tilde{J} = \frac{1}{2} \int\limits_{\Omega} (\text{grãd } \tilde{\xi})^2 d\tau \text{'}$$

Obviously the minimum is attained for grãd $\tilde{\xi}$ = 0. This implies that dĩv grãd $\tilde{\xi}$ = div D^{-2} grad ξ = 0 and guarantees because of Eq. (177) that the original differential equation (Eq. (173)) is satisfied. With Eq. (107) one obtains in the original coordinates, except for an unessential factor

$$J = \frac{1}{2} \int (D^{-1} \text{ grad } \xi)^2 d\tau \tag{178}$$

We form the variation δJ due to a variation $\delta\phi$. One has

$$\delta J = \int\limits_{\Omega} (D^{-1} \text{grad } \xi \cdot D^{-1} \text{grad } \delta\xi) d\tau$$

In order to express $\delta\xi$ by $\delta\phi$, an integration by parts is carried out

$$\delta J = \int\limits_{\Gamma} \xi(D^{-2} \text{grad } \delta\xi \cdot \vec{e}_n) do - \int\limits_{\Omega} \xi \text{ div}(D^{-2} \text{grad } \delta\xi) d\tau$$

It follows from Eq. (177) that

$$\text{div } D^{-2} \text{grad } \delta\xi = \text{div}(M \text{ grad } \delta\phi)$$

Then

$$\delta J = \int\limits_{\Gamma} \xi(D^{-2} \text{grad } \delta\xi \cdot \vec{e}_n) do - \int\limits_{\Omega} \xi \text{div}(M \text{ grad } \delta\phi) d\tau \tag{179}$$

$$= \int\limits_{\Gamma} \xi((D^{-2} \text{grad } \delta\xi - M \text{ grad } \delta\phi) \cdot \vec{e}_n) do + \int\limits_{\Omega} \text{grad } \xi \cdot M \text{ grad } \delta\phi d\tau$$

To express δJ in terms of $\delta\phi$ one must of course evaluate ξ, but according to the last equation one also needs at the boundary the values of $\delta\xi$ expressed by $\delta\phi$. This task is avoided by the following choice of the boundary conditions for ξ

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$$D^{-2} \text{grad } \xi \cdot \vec{e}_n - ((\vec{m}(\text{grad } \phi)) \cdot \vec{e}_n - f_2) = 0 \quad \text{on } \Gamma_2 \tag{180}$$

$$\xi = 0 \quad \text{on } \Gamma - \Gamma_2 \tag{181}$$

By Eqs. (107) and (108) one has

$$D^{-2} \text{grad } \xi \cdot \vec{e}_n = \text{const grad } \tilde{\xi} \cdot \vec{e}_n$$

Eq. (180) gives

$$(D^{-2} \text{grad } \delta\xi - M \text{ grad } \delta\phi) \cdot \vec{e}_n = 0 \quad \text{on } \Gamma_2 \tag{182}$$

One therefore obtains

$$\delta J = \int_\Omega M \text{ grad } \xi \cdot \text{grad } \delta\phi \, d\tau \tag{183}$$

This equation will be the starting point of the numerical work. To gain some further theoretical insight we derive the Euler equations of the variational problem. They will be satisfied if the minimum has been found, but in the methods discussed here they will not be used in the determination of the minimum. Carrying out a further integration by parts of Eq. (183) one obtains

$$\delta J = \int_{\Gamma - \Gamma_1} (M \text{ grad } \xi \cdot \vec{e}_n) \delta\phi \, d\sigma - \int_\Omega \text{div}(M \text{ grad } \xi) \delta\phi \, d\tau = 0 \tag{184}$$

The restriction of the surface integral to $\Gamma - \Gamma_1$ is possible because of Eq. (148). This gives the Euler equation

$$\text{div}(M \text{ grad } \xi) = 0 \tag{185}$$

with the boundary condition

$$M \text{ grad } \xi \cdot \vec{e}_n = 0 \quad \text{on } \Gamma - \Gamma_1 \tag{186}$$

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in addition one has Eq. (181) viz.

$$\xi = 0 \quad \text{on} \quad \Gamma - \Gamma_2 \qquad\qquad (187)$$

For an interpretation we remember that the function ξ is related to the residual in Eq. (173) by Eq. (177). The system of equations for ϕ can be regarded as a system for ξ. The partial differential equation for ξ is the differential equation for ϕ linearized for the vicinity of some solution. Elliptic and hyperbolic regions of the original problem are again elliptic and hyperbolic regions in the differential equation for ξ.

In a purely subsonic problem one obtains boundary conditions for ξ which arise by a linearization of Eqs. (174) and (175), and by replacing the perturbation of ϕ by ξ. Here $\Gamma - \Gamma_2 = \Gamma_1$ and $\Gamma - \Gamma_1 = \Gamma_2$. For a purely supersonic flow in a channel we obtain the same situation as in the preceding section. The entrance cross section belongs to Γ_1 and Γ_2, therefore, no boundary conditions for ξ are encountered. The exit cross section belongs neither to Γ_1 nor to Γ_2. Both conditions Eqs. (186) and (187) must be imposed.

For the flow in a divergent channel which goes from supersonic to subsonic speeds by means of a shock, one has a situation analogous to that of Section IV. To guarantee that $\xi = 0$ in the subsonic region one needs boundary conditions for ξ along the entire boundary of the subsonic region. This means that some homogeneous boundary condition must be provided at the shock. If such a condition is given, then because of the continuity of ξ one has on the upstream side of the shock $\xi = 0$, and because of the differential equation (Eq. (185)) (interpreted in the sense of a weak solution) also a condition for a certain derivative. This together with the boundary conditions at the side walls ensure that $\xi = 0$ in the supersonic region. There is however a difference. In the preceding sections it was possible to express the residual directly in terms of local data for the current approximation to the solution of the original problem. Here ξ is tied to the residual in the original differential

equation by a partial differential equation (Eq. (177)). A direct solution of the Euler equations with its boundary conditions and a formulation of a condition of this kind at the shock in terms of ϕ will, therefore, be practically impossible. As in the preceding section, the variation of the shock position leads to a condition at the shock of a different form, which replaces such a requirement. Even if the shock position is not changed, the Euler equation (Eq. (185)) interpreted in the sense of generalized functions gives

$$[M \; \text{grad} \; \xi]_-^+ \cdot \vec{e}_n = 0 \tag{188}$$

Since the matrix M upstream and downstream of the shock is not the same, this equation shows that at a shock grad ξ is in general not continuous. In order for the shock position to have no influence on J (Eq. (151)) one must, therefore, have

$$[(D^{-1} \text{grad} \; \xi)^2]_-^+ = 0 \tag{189}$$

Because of the continuity of ξ only the component of D^{-1}grad ξ normal to the shock can change. Eq. (189) is therefore equivalent with

$$[(D^{-1} \text{grad} \; \xi \cdot \vec{e}_n)^2]_-^+ = 0 \tag{190}$$

This is the desired additional condition, it will automatically be satisfied in a search procedure in which the shock position is allowed to vary.

To analyze the problems arising in the transition from a subsonic to a supersonic flow through a sonic line, we consider the flow in the lower half of a Laval nozzle (Figs. 8). Notice the relative locations of the sonic line CE, the "limiting" characteristics DC (namely the last characteristic which starting at the contour and running downstream reaches the sonic line) and the characteristic CF which is the last to start at the sonic line and not at the side walls. The contour HFDABCG is either of

the type Γ_1 or Γ_2 and there is no overlap of the two types. The exit cross section belongs neither to Γ_1 nor to Γ_2. Since the differential equation for ξ is the linearized form of the original problem, one has just enough condition along DEABC that after the minimum has been reached, $\xi = 0$ in DEABCD. This is the subsonic region and the supersonic region upstream of the limiting characteristic. The exit cross section GH belongs neither to Γ_1 nor to Γ_2 as in the purely supersonic problem. One therefore obtains

$$\xi = 0$$

on GH

$$M \text{ grad } \xi \cdot \vec{e}_n = 0$$

This together with the homogeneous boundary conditions for ξ on CG and HF guarantee that $\xi = 0$ in CGHFC. In the triangle DCF one obtains $\xi \equiv 0$, because, as a consequence of the above results, $\xi = 0$ along CF and along CD. For the determination of ξ a boundary condition along DF is, therefore, not necessary. The homogeneous boundary condition which one encounters there is of course compatible with $\xi \equiv 0$.

To carry out a search procedure by means of the method of conjugate gradients one needs a distance definition in the ϕ space. In a simple-minded procedure one might derive it from the scalar product

$$[\phi^{(1)}, \phi^{(2)}] = \int_\Omega \phi^{(1)} \phi^{(2)} d\tau$$

Denoting the gradient vector by $g(x,y,z)$, one then has

$$\delta J = \int g(x,y,z) \delta \phi(x,y,z) d\tau$$

Comparing with Eq. (184), one then finds

$$g(x,y,z) = \text{div}(M \text{ grad } \xi)$$

This, however, leads to familiar difficulties. Because of the double differentiation on the right of the last equation the function g will not lie within the space of admissible functions. In the discretized form of a finite dimensional vector space, this difficulty is again camouflaged.

The scalar product used by Bristeau and her co-workers revised for the Prandtl-Glauert coordinate distortion is given by

$$[\phi^{(1)}, \phi^{(2)}] = \int_\Omega (D^{-1} grad \ \phi^{(1)}) \cdot (D^{-1} grad \ \phi^{(2)}) d\tau$$

Then

$$\delta J = [g, \delta\phi] = \int_\Omega D^{-1} grad \ g \cdot D^{-1} grad \ \delta\phi d\tau$$

Hence with Eq. (183)

$$\int_\Omega M \ grad \ \zeta \cdot grad \ \delta\phi d\tau = \int_\Omega D^{-1} grad \ g \cdot D^{-1} grad \ \delta\phi d\tau \qquad (191)$$

On Γ_1, the gradient function g satisfies the same condition as $\delta\phi$

$$\delta g = 0 \quad \text{on } \Gamma_1 \qquad (192)$$

No restriction on g is imposed on Γ_2. Eqs. (191) and (192) are the starting points for numerical work.

To recognize the character of the equation for g we carry out an integration by parts in Eq. (191), and apply Eq. (192)

$$\int_\Gamma \delta\phi [D^{-2} grad \ g \cdot \vec{e}_n] do - \int_\Omega div(D^{-2} grad \ g)\phi d\tau$$
$$= \int_\Gamma \delta\phi [M \ grad \ \zeta \cdot \vec{e}_n] do - \int_\Omega div(M \ grad \ \zeta) d\tau$$

Hence

$$(D^{-2} grad \ g) \cdot \vec{e}_n = (M \ grad \ \zeta) \cdot \vec{e}_n \quad \text{on } \Gamma - \Gamma_1 \qquad (193)$$

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$$\text{div}(D^{-2}\text{grad } g) = \text{div}(M \text{ grad } \zeta) \qquad (194)$$

This together with Eq. (192) shows that g is defined except for the Prandtl-Glauert distortion by a Poisson equation with classical boundary conditions.

Next, we study the effect of shocks. First let us disregard the variation of the shock location. For a general element of the function space, M grad ζ will be discontinuous at the shock, div(M grad ζ) will give a δ-function. At a shock one has a jump in grad ζ while ϕ and ζ are continuous. Approximating ϕ (and ζ) by a sequence of continuous functions with continuous gradient and (conceptually) determining the minima of J on the members of the approximating family, one automatically takes into account in the limit of a discontinuous gradient the conditions which prevail at the shock. To obtain good results the approximating function must be sufficiently flexible. Practically this means that one must use a rather fine grid. This is the method usually employed.

For completeness we also examine the condition if one identifies the shock in all iteration steps. Let the shock location be given by

$$x = f_s(y,z)$$

In such a procedure the shock surface is considered as a cut in the region Ω. The region Ω so modified is denoted by Ω^1; Ω^1 changes of course from iteration to iteration. In a finite element procedure the elements adjacent to the shock will change their shape as the shock changes its position. Admissible functions ϕ are continuous and have continuous gradient in Ω^1, ϕ itself is continuous through the shock.

An element of the function space is described by the function ϕ and by the function $f_s(y,z)$. The definition of the scalar product must then include a contribution due to f_s

$$[(\phi^{(1)}, f_s^{(1)}); (\phi^{(2)}, f_s^{(2)})] \tag{195}$$

$$= \int_\Omega D^{-1} \text{grad } \phi^{(1)} \cdot D^{-1} \text{grad } \phi^{(2)} d\tau + \text{const} \int_{\Gamma_s^+} f_s^{(1)} f_s^{(2)} do$$

The integration is extended over Ω, but there is not a difference if one uses Ω^1 instead, because grad ϕ is bounded. Conceptually, the shock surface has an upstream and a downstream part. The integration in Eq. (194) is extended over the upstream part Γ_s^+.

The function ξ is determined by ϕ in the same manner as before (Eq. (177)). At the shock ξ is continuous and Eq. (177) is assumed to hold in the sense of weak equality. Therefore,

$$[D^{-1} \text{grad } \xi \cdot \vec{e}_n]_-^+ = [\vec{m} \cdot \vec{e}_n]_-^+ \tag{196}$$

J is defined as before. Then

$$\delta J = \int_\Omega (D^{-1} \text{grad } \xi \cdot D^{-1} \text{grad } \delta\xi) d\tau + \int_{\Gamma_s^+} [(D^{-1} \text{grad } \xi)^2]_-^+ \delta f_s dy dz$$

The first integral is transformed by an integration by parts in the same manner as above (see Eq. (179)). With the same boundary conditions as before, the first integral in Eq. (179) needs to be extended only over the two banks of the shock surface. But there it vanishes, for it follows from Eq. (196) that

$$[\text{grad } \delta\xi \cdot \vec{e}_n]_-^+ = [M \text{ grad } \delta\phi \cdot \vec{e}_n]$$

Thus

$$\delta J = \int_\Omega \text{grad } \xi \cdot (M \text{ grad } \delta\phi) d\tau + \int_{\Gamma_s^+} [D^{-1} (\text{grad } \xi)^2]_-^+ \delta f_s dy dz \tag{197}$$

The gradient in the function space now contains a function g_ϕ referring to the change of ϕ and a term g_{f_s} referring to the

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change of f_s. According to the definition of the scalar product (Eq. (195)), one now has

$$\delta J = [(g_\phi, g_{fs}); (\delta\phi, \delta f_s)] = \int_\Omega D^{-1} \text{grad } \phi \cdot D^{-1} \text{grad } \delta\phi d\tau$$

$$+ \text{const} \int_{\Gamma_s^+} g_{fs} \delta f_s do$$

Comparing this expression with Eq. (197) one obtains

$$\int_\Omega M \text{ grad } \xi \cdot \text{grad } \delta\phi + \int_{\Gamma_s^+} [(\text{grad } \xi)^2]_-^+ \delta f_s dydz = \tag{198}$$

$$\int_\Omega D^{-1} \text{grad } g_\phi \cdot D^{-1} \text{grad } \delta\phi + \text{const} \int_{\Gamma_s^+} g_{fs} \delta f_s$$

This equation would be the starting point of the numerical work. The Euler equations are obtained as before. One obtains

$$\text{div}(D^{-2} \text{grad } g_\phi) = \text{div}(M \text{ grad } \phi)$$

$$[g_\phi]_-^+ = 0 \quad \text{on } \Gamma_s$$

$$\text{grad } g_\phi \cdot \vec{e}_n = (M \text{ grad } \xi) \cdot \vec{e}_n \quad \text{on } \Gamma - \Gamma_1 - \Gamma_s$$

$$g_\phi = 0 \qquad \text{on } \Gamma - \Gamma_2 - \Gamma_s$$

$$[\text{grad } g_\phi \cdot \vec{e}_n]_-^+ = [M \text{ grad } \phi \cdot \vec{e}_n] \quad \text{on } \Gamma_s$$

$$g_{fs} = \text{const}[(\text{grad } \xi)^2]_-^+.$$

As far as the author sees, the method of Bristeau with or without shock identification encounters one difficulty. In carrying out the search procedure one must repeatedly evaluate the functional J and for this purpose the function ξ. This requires the solution of a Poisson equation which is a rather

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costly procedure. All other methods discussed here allow one to determine the functional J from the local values of the current approximation to the dependent variables and their derivatives.

In the discretized form of the problem one needs the derivatives of the funcational J with respect to the shape parameters. In previous formulation the author had suggested that this be done by numerical differentiation. In the present context this is not possible because each evaluation of J requires the evaluation of ξ for a different function ϕ. This, however, is not a major obstacle. Let p_j be the shape parameter which belong to the shape function x_j, and $p_{s,k}$ the shape parameter to the shape function ψ_k used in the approximation of f_s. Then from Eq. (197)

$$\frac{\partial J}{\partial p_j} = \int_\Omega (M \ grad \ \xi \cdot grad \ x_j d\tau$$

$$\frac{\partial J}{\partial p_{s,k}} = const \int_{\Gamma_s^+} [(grad \ \xi)^2]_-^+ \psi_k dydz$$

For the numerical work one needs a discretized form for ξ as well as ϕ. One will probably use the same shape functions x for ξ as will ϕ.

$$\xi = \Sigma \ p_{\xi,j} \ x_j$$

$$\phi = \Sigma \ p_{\phi,j} \ x_j$$

We assume that admissible functions ϕ and ξ satisfy the pertinent boundary condition. Eq. (177) is satisfied by minimizing

$$\int [D^{-1} grad \ \xi - \vec{m}(grad \ \phi)]^2 d\tau$$

with respect to variations of ξ. This gives

$$\int_\Omega [\Sigma \ p_{\xi,j} \ D^{-1} grad \ x_j - \vec{m}(\Sigma_k \ p_{\phi,k} \ x_k)]^2 d\tau = min$$

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Hence by varying $p_{\xi,\ell}$

$$\sum_j p_{\xi,j} \int_\Omega D^{-1} \text{grad } \chi_j \cdot \text{grad } \chi_\ell d\tau =$$

$$\text{for all } \ell \qquad (199)$$

$$\int_\Omega (D^{-1} \vec{m}(\sum_k p_{\phi,k} \ \chi_\ell)) \cdot \text{grad } \chi_\ell) d\tau$$

For given $p_{\phi,k}$ one can evaluate $\vec{m}$ and

$$\int_\Omega (D^{-1} \vec{m}(\sum_k p_{\phi,k} \ \chi_k)) \cdot \text{grad } \chi_\ell) d\tau.$$

The evaluation is not too time consuming because grad χ_ℓ has finite support. One obtains a sparse linear system with predetermined coefficients for the parameters $p_{\xi,j}$.

Also the gradient function g_ϕ is expressed by means of shape functions χ_ℓ. One obtains from Eq. (198)

$$\sum_j p_{g,j} \int_\Omega D^{-1} \text{grad } \chi_j \cdot D^{-1} \text{grad } \chi_\ell d\tau =$$

$$\qquad (200)$$

$$\int_\Omega (M \text{ grad } \xi \cdot \text{grad } \chi_\ell) d\tau$$

The matrix M depends upon the current approximation to ϕ, grad ξ is expressed by the pertinent coefficients $p_{\xi,k}$. This suffices to evaluate the right-hand side. The matrix on the left is identical with that of Eq. (199).

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SECTION VI
SOME GENERAL DISCUSSIONS

The formulations of the concept of residual minimization are not yet complete. Still to be expressed is the effect of a circulation in the two-dimensional case and of a wake in the three-dimensional flows, and connected with it the formulation of the Kutta condition and of far field conditions. In addition, we shall discuss the question of damping in the supersonic region and of biasing against expansion shocks. In this section we make a number of general observations to provide a background for the treatment of these questions.

Methods of residual minimization have the advantage of guaranteed convergence. Of course, if an error has been in the formulation, which does not allow a solution to exist, then the expressions obtained by the minimization procedure (i.e., the solutions of the Euler equations) will fail to satisfy the differential equations; in spite of the fact that the minimization procedure converges. Other approaches in contrast would display some warning sign, for instance, the occurrence of an overdetermined systems of equations or the divergence of an iterative approach. On the other hand, if the formulation is such that the solution is not unique, as for instance in a two-dimensional flow past a smooth contour for which the circulation remains undetermined or in flows with a supersonic region where expansion as well as compression shocks may appear, then the function J will vanish not at a single point but along some curve of the function space under consideration. (If one thinks of a contour map, with the value of J representing the height, then one has in the function space a valley with constant height zero.) In this case the search procedure will give as a result some point of the valley, which depends upon the starting point of the search. In other approaches one will then obtain a degenerate system. In such cases one must include in the expression for the functional J additional terms which increase the functional for solutions that fail to satisfy some additional

condition, for instance, a preassigned location of the rear stagnation point in the two-dimensional flow without sharp trailing edge. In two-dimensional incompressible or compressible subsonic flows one can make a conformal mapping of the profile on a smooth contour, for instance on a circle. The Kutta condition (no flow around the trailing edge) then expresses itself by the requirement that at the map of the trailing edge the velocity tangential to the profile be zero. The essential features of boundary value problems with a Kutta condition can, therefore, be discussed for a smooth contour with assigned position of the rear stagnation point. In this form it is easier to visualize flows which fail to satisfy this substitute Kutta condition. One notices that the Dirichlet integrad $\int_\Omega (\text{grad } \phi)^2 dxdy$ is invariant

under a conformal mapping. In the transformed plane with a smooth contour the integral is finite even if the substitute Kutta condition is not satisfied, the integral in the original flow plane is therefore finite, too, although grad ϕ tends to infinity. Integrals of this kind occur in the ϕ, ψ formulation, where one minimizes

$$\int (\vec{m}(\text{grad } \phi) - \vec{e}_x \psi_y + \vec{e}_y \psi_s)^2 dxdy$$

and in the Bristeau approach where one considers

$$\int_\Omega (\text{grad } \zeta)^2 dxdy$$

Because of the nonlinearity of the problem the analogy is not perfect. The argument does not apply to the w representation. $\int (\text{div } w)^2 dxdy$ and $\int (\text{curl } w)^2 dxdy$ are not invariant, and actually give an infinite contribution in the vicinity of a sharp trailing edge. This disturbing phenomenon is alleviated by the discretization process. If one should encounter difficulties, one might also reduce the influence of such a region by the use of a positive weight function which tends to zero at a sharp trailing edge.

Let use now discuss the example of a two-dimensional flow in a wind tunnel with fixed walls over a smooth two-dimensional body (Fig. 9). We prescribe the normal component of the mass flow vector at the entrance cross section and at the side walls (where it is zero). At the exit cross section we prescribe either the tangential component of the velocity, or the normal component of the mass flow vector. In the latter case one must make sure that the total flow through the entrance and the exit cross section are the same. At the profile one has the condition that the normal component of the mass flow vector be zero.

One can formulate this problem in a variety of ways.

1. In terms of a velocity potential with the governing equation

$$\text{div}(\vec{m}(\text{grad } \phi) = 0$$

2. In terms of the stream function ψ. Here one must consider the velocity vector as a function of grad ψ, $\vec{w}=\vec{w}(\text{grad } \psi)$. Then one has the governing equation

$$\text{curl } \vec{w}(\text{grad } \psi) = 0$$

(This representation is rarely used because $\vec{w}(\text{grad } \psi)$ is hard to evaluate and double-valued: for a given grad ψ one obtains one supersonic and one subsonic velocity.)

3. Simultaneous representation in terms of ϕ and ψ with the governing equations

$$(\vec{m}(\text{grad } \phi))_1 - (\text{grad } \psi)_2 = 0$$

$$(\vec{m}(\text{grad } \phi))_2 + (\text{grad } \psi)_1 = 0$$

There exist a pair of analogous equations in a formulation in which one considers $\vec{w}$ as a function of grad ψ. This possibility will not be discussed here. In the last formulation one has a
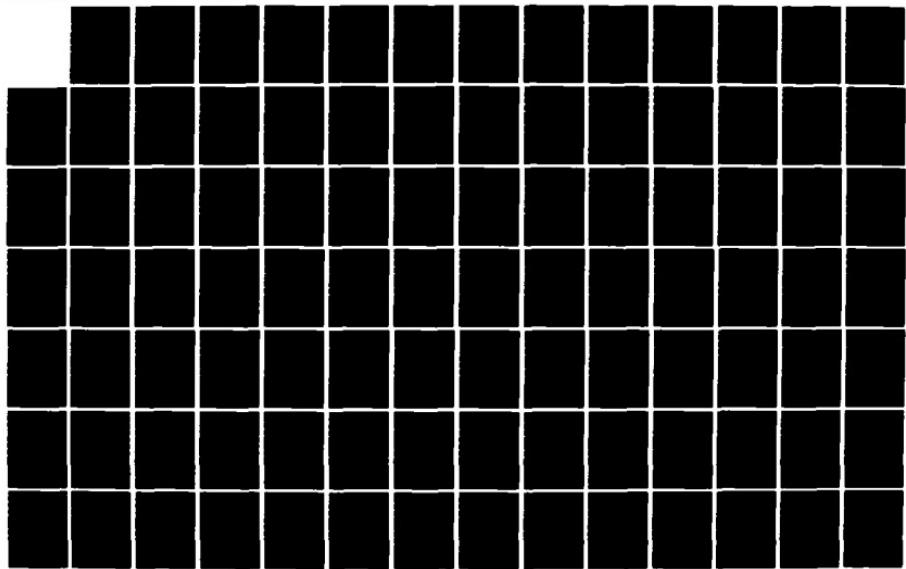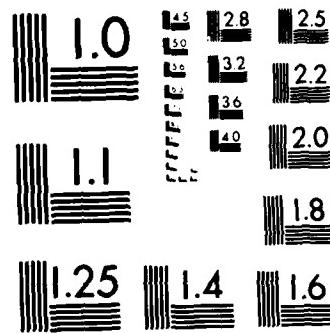
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vorticity free field defined by grad ϕ and a divergence free field defined by ψ. Both grad ϕ and grad ψ define a mass flow vector $\vec{m}$. The above equation postulates that the vectors $\vec{m}$ so defined be identical.

4. One has a representation solely in terms of $\vec{w}$ with the governing equations

$$\text{div } \vec{m}(\vec{w}) = 0$$

$$\text{curl } \vec{w} = 0$$

For such simple flow one can find the conditions that are to be imposed by inspection, using as a guide the analogy to the incompressible flow.

In a formulation in terms of ψ, we prescribe as boundary condition at the exit cross section the normal component of the mass flow. Then one can find ψ along the entire contour by an integration. Because of the postulate that the mass flow through the entrance and the exit cross section be the same, ψ will be single valued. A constant of integration can be chosen arbitrarily, for instance equal to zero at some suitable point of the contour. At the profile ψ is constant, but at this stage this constant remains undetermined. To see this, assume that the normal component of the mass flow through the entrance and the exit cross section is everywhere zero. Then one has $\psi = 0$ along the entire outer contour. Nevertheless, one can have a flow around the profile. It is similar to a potential vortex, but the vortex strength is left open. The difference in ψ between the profile and the wall gives the mass flow in a cross section extending from the profile to the wall. In a general flow pattern such a flow is superimposed to the flow through the tunnel. The constant assigned to ψ at the profile is related to the circulation around the profile (although in this formulation the concept of circulation does not appear). If there is a flow through the tunnel, one can satisfy a substitute Kutta condition (viz., that the value of the velocity at some

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given point of the smooth contour be zero) by the choice of the value of ψ at the profile.

In the formulation in terms of ϕ we assume that along the exit cross section the tangential component of the veloctiy is given. One can then determine the values of ϕ in the exit cross section by an integration. In the present discussion the specific boundary conditions in the exit cross section are unimportant; in practice the choice of the potential insures that there is no discrepancy between the mass flow in the entrance and in the exit cross section. To satisfy a substitute Kutta condition one must admit a circulation. The velocity $\vec{w} = \text{grad } \phi$ is, of course, single valued, but $\int \text{grad } \phi \cdot d\vec{s}$ formed along a curve surrounding the profile need not be single valued. Admissible functions ϕ must, therefore, be represented by an arbitrary single-valued function of x and y and added to it some standard function of x and y multiplied by an undetermined constant which increases by a constant as one makes any complete circuit around the profile. This standard function need not satisfy the flow equations (because of the nonlinearity of the problem this is impossible in any case). This second contribution is determined by the circulation, which in this case is chosen in such a manner that the substitute Kutta condition is satisfied.

In the ϕ, ψ formulation one has as conditions at the outer contour either the values of ϕ or ot ψ. If one does not impose a substitute Kutta condition but a circulation, then one must allow the value of ψ at profile to vary. If one assigns the value of ψ at the profile (while at some point of the outer contour ψ is fixed) then one must allow for some circulation.

If a substitute Kutta condition is imposed then one must admit variations of the circulation and of the value of ϕ at the profile. If one fails to do so and yet imposes a substitute Kutta condition then one will obtain an erroneous result; in spite of the fact that the solution process converges.

In the $\vec{w}$ representation neither the mass flow nor the circulation are directly expressed. As a matter of fact, for a

general element of the function space (i.e., for a general $\vec{w}$ neither the mass flow through a cross section extending from the profile to the wall nor the values of the circulation determined for different paths will be constant). Constancy of these quantities will be achieved only after the governing equations are satisfied.

In the original flow field the Kutta condition is represented by the postulate that the particles leave the trailing edge in the direction of the bisector of the angle between the upper and lower surface of the profile. (In the three-dimensional case, it is permissible that the velocity components in the direction tangential to the trailing edge are different. This is the reason for the occurrence of trailing vortices.) In a finite element representation of the flow field the Kutta condition is expressed by a linear relation between the shape parameters of some of the elements lying in the vicinity of the trailing edge.

Some further insight is gained from the three-dimensional problem. There one has a wake consisting of a sheet of trailing vortices. In an isoenergetic and isentropic flow one can still introduce a velocity potential, but it has a jump as one passes through the vortex sheet. According to the Helmholtz vorticity theorem, the circulation integral $\int \vec{w} \cdot d\vec{s}$ formed along a closed line which moves with the particles remains unchanged in time. For all curves in a flow field which can be traced back in time to curves which are free of circulation, the circulation will remain zero, no jump of the potential will be found. The only curves which cannot be traced back in this manner are those which initially began and ended at the profile or in three dimensions at the wing. It follows that the wake (where the potential jumps) is formed exclusively from particles which initially were found at the surface of the wing. If one studies the flow in the vicinity of a stagnation point formed at a smooth body, one finds that it requires an infinite time for particles at the surface to reach the stagnation point; they cannot escape from the surface. Therefore no wake is formed (theoretically) for such a body. A

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sharp trailing edge permits particles at the surface to escape
and to form a wake. (The author considers this view preferable
to one according to which the trailing edge generates the wake.)

One can imagine a smooth body, with a slight deformation so
that a sharp trailing edge arises along some preassigned line.
The flow field as a whole will remain initially unchanged. But a
wake will establish itself, and one obtains the substitute Kutta
condition mentioned before. The time needed for a particle to
escape from the surface will increase as one makes this
artificially trailing edge smaller, and it takes longer and
longer (theoretically) to establish the wake. This explains that
without such a modification one will not obtain a wake; for an
infinitesimally small modification of this kind, the time to
establish a wake would be infinite.

Conceptually, the boundary value problem for a flow with a
wake can be described as follows. One replaces the wake by an
inner boundary. For the flow field with the wake excluded in
this manner one has the usual boundary value problem for the
velocity potential. For this field one has as boundary condition
along the two banks of the wake that the velocity component
normal to it be zero. The criterion whether the assumed shape of
the wake is correct arises from the requirement that there is no
jump of the pressure. Properly speaking this does not guarantee
that the jump of the potential is the same everywhere. In any
case, the wake is assumed from the very beginning to start at the
sharp trailing edge or along the line which is defined as the
incipient trailing edge in a quasi Kutta condition at a smooth
body. No further condition is required along such a line.

The two-dimensional problem can be discussed in the same
manner. For example, one might assign the shape of the wake and
then check whether the assumed wake satisfies the requirement of
pressure equality. But in two dimensions pressure equality
amounts to an agreement of the tangential components of the
velocity on both sides of the wake, and this gives a constant
jump of the potential all along the wake. Furthermore, the jump

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of the potential can then be shifted to some curve which does not necessarily begin at the trailing edge. The two-dimensional formulation takes advantage of these simplifications and, therefore, encounters the additional condition at the trailing edge.

SECTION VII
A SIMPLIFIED EULER CODE

It is quite likely that the idea of residual minimization can be extended to include Euler codes. One has to take into account more independent variables and more unknowns. In the present report we do not explore this possibility, but ask whether a simplified version is sufficient for technical applications and what modifications of the methods previously discussed are necessary for this purpose.

In comparison to full potential codes Euler codes have two advantages: (1) they take the entropy differences which arise in a shock properly into account, and (2) they allow wake capturing (in three-dimensional problems and perhaps in two-dimensional problems with a jet flap). To the author of this report the first property seems to be of only minor importance. In most applications only weak shocks are admissible so that the entropy differences are negligible. Of course it may be desirable to verify this assertion, and for this purpose one would need a code which is able to take entropy differences into account. The possibility of wake capturing, however, is a very interesting extension which may well be important. In the following discussion we assume that the entropy is constant throughout the flow field. Moreover, in all aerodynamic application the total enthalpy $(i + w^2/2)$ is the same for all the streamlines. Then one can express the thermodynamic state in terms of the velocity vector. The velocity components are, therefore, the only dependent variables.

The equations underlying the Euler code are of course well known. They are recapitulated here in order to show the interconnection of different formulations. One starts with the conservation laws of mass, momentum and energy, and, of course, the thermodynamic equation of state for the gas. To avoid tensor notation some of the equations are written in coordinates with velocity components u, v, and w.

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$$\text{div}(\rho\vec{w}) = 0 \quad \text{(conservation of mass)} \qquad (201)$$

$$\text{div}(\rho\vec{w}(i + \tfrac{\vec{w}^2}{2}) = 0 \quad \text{(conservation of energy)} \qquad (202)$$

$$\frac{\partial}{\partial x}(p + \rho u^2) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uw) = 0$$

$$\frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(p + \rho u^2) + \frac{\partial}{\partial z}(\rho vw) = 0 \quad \text{of momentum)} \qquad (203)$$

(conservation

$$\frac{\partial}{\partial x}(\rho uw) + \frac{\partial}{\partial y}(\rho vw) + \frac{\partial}{\partial z}(p + \rho w^2) = 0$$

$$p = p(\rho, s) \qquad \text{(equation of state)} \qquad (204)$$

Here i is the specific enthalpy. Let s be the specific entropy, and T the temperature. One has

$$Tds = di - \rho^{-1}dp \qquad (205)$$

Using the relation

$$\text{div}(a\vec{b}) = a\,\text{div}\,\vec{b} + \text{grad}\,a \cdot \vec{b}$$

One obtains from Eq. (202)

$$(i + \frac{w^2}{2})\text{div}(\rho\vec{w}) + \rho\vec{w} \cdot \text{grad}(i + \frac{w^2}{2}) = 0$$

Hence with Eq. (201)

$$\vec{w} \cdot \text{grad}(i + \frac{\vec{w}^2}{2}) = 0 \qquad (206)$$

The expression $(i + \frac{\vec{w}^2}{2})$ is constant along the streamlines. If the constant is the same for all streamlines, then the flow is called isoenergetic.

The Euler equations (which are frequently taken as an alternative starting point) are obtained by subtracting Eq. (201) multiplied by $\vec{w}$ from the momentum equations.

This gives (after division by ρ)

$$\rho^{-1}(\partial p/\partial x) + u(\partial u/\partial x) + v(\partial u/\partial y) + w(\partial u/\partial z) = 0$$

$$\rho^{-1}(\partial p/\partial y) + u(\partial v/\partial x) + v(\partial v/\partial y) + w(\partial v/\partial z) = 0 \qquad (207)$$

$$\rho^{-1}(\partial p/\partial z) + u(\partial w/\partial x) + v(\partial w/\partial y) + w(\partial w/\partial z) = 0$$

These equations can be interpreted as expressions of Newton's law.

Multiplying the Euler equations respectively by u, v, and w one obtains

$$\vec{w} \cdot (\frac{1}{\rho} \, \text{grad} \, p + \text{grad} \, \frac{\vec{w}^2}{2}) = 0$$

Subtracting Eq. (206) and using Eq. (205) one obtains

$$T \, \vec{w} \cdot \text{grad} \, s = 0$$

The entropy is therefore constant along streamlines (except for shocks). If the constant is the same for all streamlines, then the flow is called isentropic.

Consider now an isoenergetic, isentropic flow. Subtracting from the respective Euler equations

$$\text{grad}(i + \frac{\vec{w}^2}{2}) = 0 \qquad (208)$$

and taking into account that ds = 0, and therefore

$$di = \rho^{-1}dp$$

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one finds

$$v\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + w\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = 0$$

$$u\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + w\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) = 0 \qquad (209)$$

$$u\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial y}\right) + v\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = 0$$

Now

$$\mathrm{curl}\ \vec{w} = \vec{e}_x\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \vec{e}_y\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \vec{e}_z\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

The last equations can then be written

$$\vec{w} \times \mathrm{curl}\ \vec{w} = 0 \qquad (210)$$

In an isentropic, isoenergetic flow the velocity vector has the direction of the streamlines. In other words, stream tubes and vortex tubes coincide. Since for any $\vec{w}$

$$\mathrm{div}\ \mathrm{curl}\ \vec{w} = 0$$

it follows that if the vorticity vector is zero somewhere along a streamline it is zero along the entire streamline. This holds in most aerodynamic applications. In a flow without vorticity one can introduce a velocity potential

$$\vec{w} = \mathrm{grad}\ \phi$$

Then, because the flow is isoenergetic

$$i + (1/2)(\mathrm{grad}\ \phi)^2 = \mathrm{const} \qquad (211)$$

Eq. (210) or its representation in components Eqs. (209) and also Eq. (208) and from there going back the Euler equations (Eqs. (207)) are then satisfied. One has the important result: if one

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introduces a potential and expresses the pressure by means of Eq. (211) (usually called Bernoulli's equation), then Euler's equations of motion are satisfied. In this chain of equations the equation of conservation of mass is <u>not</u> involved. The density can be expressed in terms of the potential by means of the Bernoulli equation. Expressing all data in the equation for conservation of mass in terms of the velocity potential one obtains the familiar potential equation.

The author finds the observation that the Euler equations are automatically satisfied by the introduction of the potential, and that the potential is ultimately determined from the equation of conservation of mass attractive because of its simplicity. Of course conservation of momentum can be obtained by combining the Euler equations with the equations for conservation of mass, and in this sense the potential equation serves to satisfy the momentum equations. But this interpretation (occasionally put forth by Jameson) is rather indirect.

The only place in a flow where vorticity can arise is the wake; all other streamlines start at *infinity with zero vorticity* and, therefore, retain the zero vorticity. But the streamlines of the wake start at the trailing edge and therefore nothing can be said about their vorticity.

The local character of the wake flow can be visualized as follows: Consider a plate in the plane $y = 0$ which extends from upstream infinity to the z-axis. Assume that one has on the upper side a velocity vector $\vec{e}_x c_1 + \vec{e}_z c_2$ and on the lower side $\vec{e}_x c_3 + \vec{e}_z c_4$ and that $c_1^2 + c_2^2 = c_3^2 + c_4^2$. Between these two flows there is no pressure difference. Starting at the z-axis one has a vortex sheet caused by the difference in the z velocity. The vorticity vector has the character of a delta function. In a two-dimensional flow this phenomenon does not occur, because the z components are zero.

If one describes the flow field by means of a potential, then one must obviously exclude the wake. Such a description is possible if one identifies the vortex sheet and considers it as a

cut in the flow field. One then imposes the condition that the velocity components normal to the vortex sheet be zero and that the absolute value of the velocity vector on the upper and lower side be the same. The shape of the vortex sheet and the requirement that the velocity components normal to it be zero together with the other boundary conditions of the flow field determine the potential flow completely. If the shape of assumed vortex sheet is incorrect, then there will be a pressure difference between the two sides of the sheet. In a method of residual minimization using the potential one must include terms for the failure to satisfy this condition. The element will contain beside the function φ also a function describing the shape of the vortex sheet.

In a two-dimensional flow there is no discontinuity of $\vec{w}$ and the identification of the wake during the computation is not necessary. If one works with $\vec{w}$ (and curl $\vec{w}$ = 0), then no extra provisions are needed. If one works with the potential, then one must introduce a line along which the potential is allowed to jump by an unknown constant. This line need not be identified with the wake. The jump of the potential must not be interpreted as an infinite gradient. The gradient of φ is continuous as one passes through this line of discontinuity.

Let us now ask how one can formulate the equations for an isentropic, isoenergetic flow so that one obtains a wake capturing method. The pressure is determined everywhere, by the velocity vector, even in the wake if one assigns to it a non-vanishing thickness. In the discretized form of the problem a jump of the velocity between the upper and lower side of the wake is distributed over at least one mesh. The original wake appears as a layer of finite thickness. In this layer one has vorticity, but because of the assumptions of isentropic, isoenergetic flow it is subject to Eq. (210)

$$\vec{w} \times \text{curl } \vec{w} = 0.$$

To make wake capturing possible in the $\vec{w}$ formulation, it suffices if one replaces the original condition

$$\text{curl } \vec{w} = 0$$

by the last equation. Of course, this term is used also in the definition of J. Only in the region where one expects to find the wake is this modification needed.

SECTION VIII
FAR FIELD CONDITIONS, BASIC FORMULAE

Far field conditions are of interest because they allow one to reduce the size of the computed part of the flow patterns. They can be included in the approach by residual minimization without difficulty. Of course, in steady problems they are only of minor importance because the effect of errors introduced at the far boundaries are fairly small in the vicinity of a wing. In preliminary tests of the method they may therefore be disregarded.

Let us first recapitulate the general concepts. Far field conditions serve to match the computed part of the flow field at its outer boundary with the "distant" (not computed) flow field. In the distant field the flow equations must be satisfied and the solutions that are used must die out as one travels to infinity. In order to express this behavior, it is necessary that in the distant flow field the flow differential equations can be solved analytically. In the distant field the flow is close to a parallel flow with the assigned free stream Mach number. There- fore, the linearized flow equations for compressible flow are applicable. After having carried out the Prandtl-Glauert coordinate transformation one deals with the Laplace equation. There fundamental solutions which have the desired behavior at infinity are readily available. We first deal with the simpler, but less important problem of flows without circulation or trailing vortices. This provides an easier access to the general ideas.

For the determination of the gradient function g the method of residual minimization requires within the computed part of the flow field the repeated solution of Poisson equations if one makes everywhere the Prandtl-Glauert transformation. In one method of solving these Poisson equations, one first determines a particular solution by means of integrals which use fundamental solutions (usually fundamental solution representing sources). The boundary conditions at the profile are satisfied in a next

step by solving an integral equation (in which the region of integration is given by the surface of the wing). In the inhomogeneous part as well as in the formulation of the integral equation one uses a fundamental solution which satisfy the far field conditions. In such a procedure far field conditions are, therefore, automatically satisfied. This observation has been made by Johnson and his co-workers. (One notices that the gradient function for the potential so obtained is one valued; changes of the circulation must be taken into account separately). If one employs methods which solve the Poisson equations by means of finite difference or finite element approaches, one always needs some conditions at the outer edge of the computed flow field. Otherwise the flow field remains undetermined.

The necessary number of such conditions can be seen if one considers a classical boundary value problem (in which boundary conditions different from proper far field conditions are imposed). One can for instance prescribe the potential at the outer boundary of the computed flow field. In the discretized formulation this determines certain shape parameters in the elements adjacent to the outer edge of the computed flow field. This shows in a specific case how many conditions are necessary at the outer boundary of the computed flow field. Correct far field conditions have a global character, i.e., they involve all points of the far boundary. (Such conditions are derived in Reference 3).

In steady flows these conditions can be relaxed. After one has made the Prandtl-Glauert transformation, the potential in the distant field is given (aside from the contribution of the circulation) by expressions $\mathrm{Re}\ z^{-n}$ and $\mathrm{Im}\ z^{-n}$; $n = 1,2,\ldots$ with $z = x + iy$. Terms which carry perturbations back from the outer boundary to the profile have the same form with positive exponents starting with $n = 2$. It follows that only terms with small values of n will be of importance in the vicinity of the profile. Accordingly it suffices if one represents the distant field (with which the computed part of the flow field is matched)

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aside from the underlying parallel flow and the term for the circulation, by just the first few terms of the above development. Along a ray through the origin the potential then behaves as $r^{-1}, r^{-2}, \ldots r^{-\kappa}$ in two dimensions and $r^{-2}, r^{-3}, \ldots r^{-k}$ in three dimensions. Solutions $\log r$ in the two-dimensional case and r^{-1} are not included. They represent sources and their coefficient is zero if one deals with the flow over a closed body. The dependence upon the radius is the same whether or not one carries out the Prandtl-Glauert transformation.

If the potential has this form, then it satisfies conditions first proposed in a more complicated context by Turkel and his co-workers (Reference 4). In two-dimensional flows one postulates

$$\left(\frac{2\kappa-1}{r} + \frac{\partial}{\partial r}\right)(\ldots)(\ldots)\ldots\left(\frac{3}{r} + \frac{\partial}{\partial r}\right)\left(\frac{1}{r} + \frac{\partial}{\partial r}\right)\phi = 0 \qquad (212)$$

To examine the effect of this operator on ϕ we observe that

$$\left(\frac{n}{r} + \frac{\partial}{\partial r}\right)r^{-\ell} = (n-\ell)r^{-\ell-1}$$

Therefore

$$\left(\frac{2\kappa-1}{r} + \frac{\partial}{\partial r}\right)(\ldots)(\ldots)\ldots\left(\frac{3}{r} + \frac{\partial}{\partial r}\right)\left(\frac{1}{r} + \frac{\partial}{\partial r}\right)r^{-n}$$

$$= (1-n)(2-n)\ldots(k-n)r^{-n-k}$$

The condition Eq. (212) is, therefore, satisfied for $n = 1, \ldots, k$. Solutions of the Laplace equation in two dimensions which have the same dependence upon the polar angle are given by

$$(c_1 r^{-n} + c_2 r^{+n})\exp(in\theta)$$

Applying the condition Eq. (212) to this expression at some radius $r = R_0$ one obtains

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$$\exp(in\,\theta)[c_1(1-n)(2-n)\ldots(k-n)R_0^{-n-k} +$$

$$c_2(1+n)(2+n)\ldots(k+n)R_0^{n-k}] = 0$$

Hence

$$c_2 = -c_1 \frac{(1-n)(2-n)\ldots(k-n)}{(1+n)(2+n)\ldots(k+n)} R_0^{-2n}$$

One recognizes again that for $1 \leq n \leq k$ the boundary conditions are "nonreflective." (Notice that the condition Eq. (212) contains derivatives with respect to r not with respect to the normal to the boundary.) For $n > k$ the coefficient c_2 (which is responsible for a faulty reflection) contains a factor R_0^{-2n}. For sufficiently large R_0 and n this error becomes unimportant.

For three-dimensional problems the operator corresponding to Eq. (212) is given by

$$\left(\frac{2k-2}{r} + \frac{\partial}{\partial r}\right)\ldots\left(\frac{4}{r} + \frac{\partial}{\partial r}\right)\left(\frac{2}{r} + \frac{\partial}{\partial r}\right)\phi = 0 \qquad (213)$$

These boundary conditions have an unconventional form; the derivatives which are encountered are of higher order than those occurring in the differential equation. One can use the partial differential equation and its derivatives (with respect to x, y, or z) to express derivatives with respect to r in terms of derivatives formed within the far surface of the computed region of ϕ and of its normal derivative. The far field conditions can thus be expressed by the shape parameters belonging to the elements adjacent to the far boundary. The derivatives within the far boundary are of rather high order. This can be explained by the observation that one deals with an approximation to far field conditions which have global character and, therefore, involve all points of the boundary.

The conditions Eq. (212) is applied at the points of the outer boundary of the computed field. So far we have considered only the formulation in terms of ϕ. For the formulation in terms of ϕ and $\vec{A}$ the conditions remain the same, because the

differential equation in terms of ϕ and $\vec{A}$ is easily transformed into one for ϕ alone; there are no extra conditions for $\vec{A}$. A similar situation can be expected in the $\vec{w}$ formulation. Conditions are imposed only to one of the components of $\vec{w}$. The powers of r which occur in the $\vec{w}$ formulation are $r^{-2}, r^{-3}, \ldots$ for the two-dimensional case, and $r^{-3}, r^{-4}, \ldots$ for the three-dimensional case. The conditions therefore assume a slightly different form. Details are omitted.

A formulation completely in keeping with the concept of residual minimization is obtained by considering the distant field in its entirety as a single element. The particular solutions which one wants to include in the representation of the distant field are the shape functions and their coefficients the shape parameters.

We illustrate this procedure for the Laplace equation in two dimensions using the ϕ, ψ representation. This is preferable to the representation solely in terms of ϕ, because then the auxiliary function ξ complicates the development.

The region considered is a circular ring, the inner and the outer boundaries are given by $r = 1$, and $r = R_0$, respectively; $R_0 \gg 1$. In polar coordinate one has as basic equations

$$r \phi_r - \psi_\theta = 0$$

$$\frac{1}{r} \phi_\theta + \psi_r = 0$$

(214)

It is assumed that these are the governing equations also for the computed part of the flow field (the region $1 \le r \le R_0$). The familiar form of the two-dimensional potential equation is immediately obtained from Eq. (214) by cross-differentiation and addition.

$$\frac{\partial}{\partial r}(r \phi_r) + \frac{1}{r} \phi_{\theta\theta} = 0$$

(215)

Particular solutions which have the correct form in the distant field are given by

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = r^{-n} \; \exp(in\theta) \begin{pmatrix} 1 \\ i \end{pmatrix}$$

For simplicity we restrict ourselves to solutions which are symmetric in ϕ with respect to the x-axis. Then one has in the distant field

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sum_{m=1}^{k} c_m r^{-m} \begin{pmatrix} \cos (m\theta) \\ -\sin (m\theta) \end{pmatrix}, \; r > R_0 \qquad (216)$$

The constants c_m are subject to variation. In the inner field $(1 < r < R_0)$ we set

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sum_{m=1}^{\infty} \begin{pmatrix} f_m(r)\cos(m\theta) \\ -g_m(r)\sin(m\theta) \end{pmatrix} \qquad (217)$$

Here the unknown functions $f_m(r)$ and $g_m(r)$ are determined by residual minimization. At the inner boundary the function ψ is prescribed,

$$g_m \Big|_{r=1} = b_m \qquad (218)$$

is therefore known. At the boundary $r = R_0$, the functions ϕ and ψ, taken from the regions $r \leq R_0$ and $r \geq R_0$ must match. Therefore

$$f_m(R_0) = c_m R_0^{-m}, \quad m=1,\ldots k$$

$$g_m(R_0) = c_m R_0^{-m} \qquad (219)$$

$$f_m(R_0) = 0$$
$$\qquad\qquad m > \kappa \qquad (220)$$
$$g_m(R_0) = 0$$

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There is no residual in the distant region. The expression to be minimized is given by

$$\int_{\theta=0}^{2\pi} \int_{r=1}^{R_0} [(\phi_r - \psi_\theta/r)]^2 + ((\phi_\theta/r) + \psi_r)^2] r \, dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=1}^{R_0} \{[\sum_{m=1}^{\infty} (f_m' + (mg_m/r)) \cos(m\theta)]^2 + [\sum_{m=1}^{\infty} (-(mf_m/r) - g_m') \sin(m\theta)]^2\} r \, dr \, d\theta$$

Here the integrations with respect to θ can be carried out and one obtains

$$\pi \int_{r=1}^{R_0} \sum_{m=1}^{\infty} (f_m' + (mg_m/r))^2 + ((mf_m/r) + g_m')^2 r \, dr$$

Varying f_n one obtains

$$2\pi \int_{r=1}^{R_0} \{(f_n' + ng_n/r) \delta f_n' + ((nf_n/r) + g_n')(n\delta f_n/r)\} r \, dr$$

The effect of a variation of δg_n is obtained by interchanging f_n and g_n. The usual integrations by parts yield

$$r(f_n' + ng_n/r) \Big|_{r=1}^{r=R_0} = 0$$

$$(221)$$

$$r(g_n' + nf_n/r) \Big|_{r=1}^{r=R_0} = 0$$

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$$- \frac{d}{dr}(rf_n' + ng_n) + n(nf_n/r) + g_n') = 0$$

$$- \frac{d}{dr}(rg_n' + nf_n) + n(ng_n/r) + f_n') = 0 \tag{222}$$

These equations are, of course, satisfied by

$$\begin{pmatrix} f_n \\ g_n \end{pmatrix} = r^{-n}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} f_n \\ g_n \end{pmatrix} = r^{n}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{223}$$

They are solutions of the original system Eq. (214) (see also Eq. (216)). For these expressions the terms in the parenthesis of Eq. (222) vanish separately. This, however, does not exhaust the list of possible particular solutions. Altogether the following set, which includes the particular solutions Eq. (223), is obtained

$$\begin{pmatrix} f_n \\ g_n \end{pmatrix} = r^{-n}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} f_n \\ g_n \end{pmatrix} = r^{-n}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} f_n \\ g_n \end{pmatrix} = r^{n}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} f_n \\ g_n \end{pmatrix} = r^{n}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{224}$$

Let $n > k$. Satisfying the conditions Eqs. (220) at $r=R_0$ one obtains

$$\begin{pmatrix} f_n \\ g_n \end{pmatrix} = c_1 \begin{pmatrix} (r/R_0)^{-n} - (r/R_0)^n \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ (r/R_0)^{-n} - (r/R_0)^n \end{pmatrix} \tag{225}$$

where so far c_1 and c_2 are arbitrary. Imposing the condition for $r=1$, Eq. (218), one obtains

$$c_2(R_0{}^n - R_0^{-n}) = b_n \tag{226}$$

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There is no condition for f_n at $r=1$. Accordingly $\delta f_n \neq 0$, and therefore, from the first of Eqs. (221)

$$rf_n' + ng_n \Big|_{r=1} = 0$$

Substitution of Eq. (225) gives

$$c_1(-nR_0^n - nR_0^{-n}) + c_2 n(R_0^n - R_0^{-n}) = 0$$

Hence with Eq. (226)

$$c_1 = b_n \frac{1}{R_0^n(1+R_0^{-2n})}$$

Therefore

$$\binom{f_n}{g_n} = b_n \left\{ \left(\begin{array}{cc} \frac{1}{1+R_0^{-2n}} & r^{-n}(1-(r/R_0)^{2n}) \\ 0 \end{array} \right) + \left(\begin{array}{cc} \frac{1}{1-R_0^{-2}} & 0 \\ & r^{-n}(1-(r/R_0)^{n}) \end{array} \right) \right\} \quad (227)$$

For R_0 and n large and $r \ll R_0$, one obtains the expression which would satisfy the exact conditions in the distant field, namely

$$\binom{f_n}{g_n} = b_n \, r^{-n}\binom{1}{1}$$

One can split off from the expression Eq. (227) terms of the form Eq. (223), which satisfy the original differential equation (Eq. (214)), but certain other terms which satisfy only Eq. (222) are left. The fact that the representation of the distant field is not completely general (because k is assumed to be finite) shows itself by the failure to satisfy the original differential equation exactly.

For $1 \leq n \leq k$ it follows from Eq. (219) that

$$f_n(R_0) = g_n(R_0) \quad (228)$$

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and furthermore that

$$\delta f_n(R_0) = \delta c_n R_0^{-n}$$

$$\delta g_n(R_0) = \delta c_n R_0^{-n}$$

(229)

With the last equation one obtains from Eq. (221)

$$\delta c_n [rf_n' + ng_n + rg_n' + nf_n] R_0^{-n} = 0 \tag{230}$$

Since variations of c_n are permissible, the expression in brackets must vanish. Using the particular solutions Eqs. (224) we set

$$\begin{pmatrix} f_n \\ g_n \end{pmatrix} = c_1 \begin{pmatrix} r^{-n} \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ r^{-n} \end{pmatrix} + c_3 \begin{pmatrix} r^n \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ r^n \end{pmatrix}$$

Substituting this expression into Eq. (230) one obtains

$$2nc_3 + 2nc_4 = 0$$

Accordingly

$$\begin{pmatrix} f_n \\ g_n \end{pmatrix} = c_1 \begin{pmatrix} r^{-n} \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ r^{-n} \end{pmatrix} + c_3 \begin{pmatrix} r^n \\ -r^n \end{pmatrix}$$

The condition (Eq. (228)) gives

$$c_1 R_0^{-n} + c_3 R_0^{n} = c_2 R_0^{-n} - c_3 R_0^{n}$$

Hence

$$c_3 = \frac{c_2 - c_1}{2} R_0^{-2n} \tag{231}$$

At $r=1$ one can vary f_n freely while g_n is fixed by the boundary condition. Therefore, from Eq. (221)

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$$rf_n' + ng_n \Big|_{r=1} = 0$$

Hence

$$-nc_1 + nc_3 + nc_2 - nc_3 = 0$$

Hence, $c_1 = c_2$, and, with Eq. (231), $c_3 = 0$. Thus, one obtains finally (with Eq. 218))

$$\binom{f_n}{g_n} = r^{-n} \binom{1}{1}$$

As expected, this is the exact solution for this particular value of $n \leq k$.

Next we derive the particular solutions necessary to represent the distant field. They contain one term representing the desired parallel flow and in addition the solutions of the linearized flow equation, which represent the deviations which one wants to admit in the distant field. In the $\tilde{x}, \tilde{y}$ system the potential for the perturbation terms is given by

$$\tilde{\phi} = \text{Re}(\tilde{x} + i\tilde{y})^{-2n}$$
$$, \quad n=1,2,\ldots \tag{232}$$
$$\text{and} \qquad \tilde{\psi} = \text{Im}(\tilde{x} + i\tilde{y})^{-2n}$$

Hence with

$$r^2 = x^2 + \beta^2 y^2$$

in the x,y system,

$$\phi = \text{Re}(x + i\beta y)^{-n} = r^{-2n}\text{Re}(x - i\beta y)^n$$

or $\qquad\qquad\qquad$ (233)

$$\phi = \text{Im}(x + i\beta y)^{-n} = r^{-2n}\text{Im}(x - i\beta y)^n$$

We include the term representing a circulation

$$\tilde{\phi}_\Gamma = \text{arctg}(\tilde{y}/\tilde{x})$$

$\qquad\qquad\qquad$ (234)

$$\phi_\Gamma = \text{arctg}(\beta y/x)$$

Since

$$\vec{w} = \text{grad } \phi$$

one obtains the following expressions

$$\vec{w} = -\vec{e}_x n\text{Re}(x + i\beta y)^{-n-1} + \vec{e}_y\, n\beta\text{Im}(x + i\beta y)^{-n-1}$$

or

$$\vec{w} = nr^{-2(n+1)}[-\vec{e}_x\text{Re}(x - i\beta y)^{(n+1)} + \vec{e}_y\beta\text{Im}(x - i\beta y)^{(n+1)}]$$

and similarly $\qquad\qquad\qquad$ (235)

$$\vec{w} = nr^{-2(n+1)}[-\vec{e}_x\text{Im}(x - i\beta y)^{(n+1)} - \vec{e}_y\beta\text{Re}(x - i\beta y)^{(n+1)}]$$

Specifically, for n=1

$$\vec{w} = r^{-4}[-\vec{e}_x(x^2 - \beta^2 y^2) - \beta^2\vec{e}_y 2xy]$$

and $\qquad\qquad\qquad$ (236)

$$\vec{w} = r^{-4}(\vec{e}_x 2\beta xy - \beta(x^2 - \beta^2 y^2))$$

The circulation term gives

$$\vec{w}_\Gamma = \beta r^{-2}(-\vec{e}_x y + \vec{e}_y x) \qquad\qquad (237)$$

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For each value of n there are two particular solutions. Unless one of them can be excluded by reasons of symmetry pertaining to the chosen values of n, both should be included in the representation of the flow at infinity. Stopping at n=1 and denoting the two pertinent particular solutions by $\phi^{(1)}$ and $\phi^{(2)}$ and the one for the circulation by ϕ^{Γ}, and using the same superscripts for $\vec{w}$, one then has

$$\phi = \vec{e}_x M_\infty x + c_\Gamma \phi_\Gamma + c_1 \phi^{(1)} + c_2 \phi^{(2)}$$

or

$$\vec{w} = \vec{e}_x M_\infty + c_\Gamma \vec{w}^{(\Gamma)} + c_1 \vec{w}^{(1)} + c_2 \vec{w}^{(2)}$$

(238)

The $\phi, \vec{A}$ representation simplifies in the two-dimensional case to a ϕ, ψ representation. The expressions $\tilde{\psi}$ pertaining to Eq. (232) are

$$\tilde{\psi} = \text{Im}(\tilde{x} + i\tilde{y})^{-n}$$

and

$$\tilde{\psi} = -\text{Re}(\tilde{x} + i\tilde{y})^{-n}$$

(239)

The vector $\tilde{B}'$ in the discussion following Eq. (141) is given by

$$\tilde{B}' = \vec{e}_z \cdot \tilde{\psi}.$$

For the vector $\vec{A}'$ one has

$$\vec{A}' = \vec{e}_z \psi(x,y)$$

$\vec{A}'$ and $\tilde{B}'$ are connected by the equation preceding Eq. (142), viz.

$$\rho_0 D \vec{B}' = \vec{A}'$$

(240)

Therefore, $\psi(x,y)$ is __not__ obtained by treating $\tilde{\psi}$ as a scalar and simply rewriting it in the changed coordinate system. Using the definition of D, Eq. (103) (with β defined in Eq. (101)), and the transformation formulae Eqs. (102) one obtains

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$$\psi(x,y) = \rho_0 \beta \ \mathrm{Im}(x + i\beta y)^{-n}$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (241)

$$\psi(x,y) = -\rho_0 \beta \ \mathrm{Re}(x + i\beta y)^{-n}$$

The Eqs. (233) and (241) combined satsify indeed the basic equations

$$\rho_0 \beta^2 \phi_x = \psi_y$$

$$\phi_y = -\psi_x$$

The particular solutions for the potential needed in the three-dimensional case are obtained as derivatives of all orders with respect to $\tilde{x}$, $\tilde{y}$, or $\tilde{z}$ of $\tilde{\phi} = (\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2)^{-1}$. Some of the higher derivatives are linearly dependent, because of the potential equation

$$\tilde{\phi}_{\tilde{x}\tilde{x}} + \tilde{\phi}_{\tilde{y}\tilde{y}} + \tilde{\phi}_{\tilde{z}\tilde{z}} = 0$$

The construction of these particular solutions by means of differentiation is simpler in the present case than the formulae in terms of spherical harmonics, because only the first few of these particular solutions are needed. One obtains, for instance, by forming first derivatives

$$\tilde{\phi}^{(1)} = - \ \tilde{x}\tilde{r}^{-3}, \ \tilde{\phi}^{(2)} = \tilde{y}\tilde{r}^{-3}, \tilde{\phi}^{(3)} = \tilde{z}\tilde{r}^{-3} \qquad (242)$$

We set $\mathrm{grad} \ \tilde{\phi}^{(i)} = \tilde{\vec{u}}^{(i)}$. One obtains

$$\tilde{u}^{(1)} = \tilde{r}^{-5}[\vec{e}_x(2\tilde{x}^2-\tilde{y}^2-\tilde{z}^2) + \vec{e}_y 3\tilde{x}\tilde{y} \qquad\qquad + \tilde{e}_z 3\tilde{x}\tilde{z})]$$

$$\tilde{u}^{(2)} = \tilde{r}^{-5}[\tilde{e}_x 3\tilde{x}\tilde{y} \qquad\qquad + \vec{e}_y(2\tilde{y}^2-\tilde{x}^2-\tilde{z}^2) + \tilde{e}_z 3\tilde{y}\tilde{z})] \qquad (243)$$

$$\tilde{u}^{(3)} = \tilde{r}^{-5}[\tilde{e}_x 3\tilde{x}\tilde{z} \qquad\qquad + \vec{e}_y 3\tilde{y}\tilde{z} \qquad\qquad + \tilde{e}_z(2\tilde{z}^2-\tilde{x}^2-\tilde{y}^2)]$$

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One has

$$\text{div}\ \tilde{\vec{u}}^{(i)} = \text{div grad}\ \tilde{\phi}^i = 0$$

Hence in the x,y,z system

$$\text{div}(D^{-1}\vec{u}^{(i)}) = 0 \tag{244}$$

The differential equation to be satisfied by the velocities $\vec{w}^{(i)}$ is

$$\text{div}(\beta^2 D^{-2}\vec{w}^{(i)}) = \beta^2 \text{div}(D^{-2}\vec{w}^{(i)}) = 0$$

Because of Eq. (244) it is satisfied if one sets

$$\vec{w}^{(i)} = D\vec{u}^{(i)}$$

Hence from Eq. (243)

$$\vec{w}^{(1)} = r^{-5}[\vec{e}_x(2x^2-\beta^2(y^2+z^2))+\vec{e}_y 3\beta^2 xy \qquad\qquad +e_z 3\beta^2 xz]$$

$$\vec{w}^{(2)} = r^{-5}[\vec{e}_x 3\beta xy \qquad\qquad +\vec{e}_y\beta(2\beta^2 y^2-x^2-\beta^2 z^2)+\vec{e}_z 3\beta^2 yz]$$

$$\vec{w}^{(3)} = r^{-5}[\vec{e}_x 3\beta xz \qquad\qquad +\vec{e}_y 3\beta^2 yz \qquad\qquad +\vec{e}_z\beta(2\beta^2 z^2-x^2-\beta^2 y^2]$$

To determine the function $\vec{A}$ pertaining to ϕ we use the formulation following Eq. (141). In the $\tilde{x},\tilde{y},\tilde{z}$ system $\tilde{\phi}$ is connected with an auxiliary vector field $\vec{B}$ by

$$\tilde{\text{curl}}\ \tilde{\vec{B}} = \tilde{\text{grad}}\ \tilde{\phi} \tag{245}$$

Consider the function $\tilde{\vec{\phi}}^{(1)}$ defined in Eq. (242). One has

$$\tilde{\phi}^{(1)} = \tilde{x}\tilde{r}^{-3} \tag{246a}$$

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and

$$\text{grad } \tilde{\phi}^{(1)} = \tilde{r}^{-5}[\vec{e}_x(2\tilde{x}^2-\tilde{y}^2-\tilde{z}^2) + \vec{e}_y 3\tilde{x}\tilde{y} + \vec{e}_z 3\tilde{x}\tilde{z}]$$

With the definition

$$\text{curl } \tilde{B} =$$

$$\vec{e}_x((\partial\tilde{\vec{B}}_3/\partial\tilde{y})-(\partial\tilde{\vec{B}}_2/\partial\tilde{z}))+\vec{e}_y((\partial\tilde{\vec{B}}_1/\partial\tilde{z})-(\partial\tilde{\vec{B}}_3/\partial\tilde{x}))+\vec{e}_z((\partial\tilde{\vec{B}}_2/\partial\tilde{x})-(\partial\tilde{\vec{B}}_1/\partial\tilde{y}))$$

one finds that Eq. (245) is satisfied by

$$\tilde{\vec{B}}^{(1)} = \tilde{r}^{-3}(\vec{e}_z\tilde{y} - \vec{e}_y\tilde{z}) \tag{246b}$$

Similarly

$$\tilde{\phi}^{(2)} = -\tilde{y}\tilde{r}^{-3}$$

$$\tilde{\vec{B}}^{(2)} = \tilde{r}^{-3}(\vec{e}_x\tilde{z} - \vec{e}_z\tilde{x}) \tag{247}$$

and

$$\tilde{\phi}^{(3)} = -\tilde{z}\tilde{r}^{-3}$$

$$\tilde{\vec{B}}^{(3)} = \tilde{r}^{-3}(\vec{e}_y\tilde{x} - \vec{e}_x\tilde{y}) \tag{248}$$

Transferring these expressions into the x,y,z system one obtains, with $r^2 = x^2 + \beta^2(y^2 + z^2)$,

$$\phi^{(1)} = -xr^{-3}$$

$$\vec{B}^{(1)} = r^{-3}\beta(\vec{e}_z y - \vec{e}_y z)$$

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and according to Eq. (240) (using the definitions Eqs. (103) and Eqs. (101))

$$\vec{A} = \rho_0 r^{-3} \beta^2 (\vec{e}_z y - \vec{e}_y z)$$

Similarly

$$\phi^{(2)} = -\beta y r^{-3}$$

$$\vec{A}^{(2)} = \rho_0 r^{-3} \beta (\vec{e}_x z - \vec{e}_z x)$$

and

$$\phi^{(3)} = -\beta z r^{-3}$$

$$\vec{A}^{(3)} = \rho_0 r^{-3} \beta (\vec{e}_y x - \vec{e}_x y)$$

Further particular solutions can be obtained by differentiations with respect to x, y, or z.

These formulae enable one to determine the shape functions for the element which forms the distant field; in particular their values at the far boundary of the computed field. In the interior of the computed field one shape function has as support several elements which are adjacent to each other. In the same manner the shape functions, which for the distant field are given by the expressions shown above, have contributions in all the elements of the computed field adjacent to the far boundary. In these elements they are, of course, expressed by means of elemental shape functions of the same character as all other elemental shape functions in the computed part of the flow field.

SECTION IX

FAR FIELD CONDITIONS AND CIRCULATION EFFECTS
IN TWO-DIMENSIONAL FLOWS

The presence of circulation in a two-dimensional flow has an effect rather different from the presence of a vortex sheet in three dimensions. The circulation in a two-dimensional flow is described by just one parameter. The particular solution representing the circulation in the distant field satisfies the linearized equation for the distant field exactly. The vortex sheet in the distant field of a three-dimensional flow constitutes a much more concentrated perturbation of complicated character, so that one can use even in the distant field only an approximate representation. The two cases are, therefore, treated separately.

In the two-dimensional case, it seems as if it suffices, if one includes in the representation of the distant field the one term which gives a circulation. If one uses a representation which uses the idea of a potential, then it is continued into the computed part of the flow field by an expression which gives after one circuit around the profile constant jump of the potential. In the $\vec{w}$ representation, it is not necessary to include such a term.

The $\vec{w}$ and the ϕ,ψ methods have in common that the integrals J that are to be minimized contain in the integrand directly the governing equation. But in the distant field these equations (in their linearized form) are exactly satisfied. The integrals J therefore need to be evaluated only over the computed part of the flow field. They still depend upon the parameters of the distant field because the support regions of the shape functions pertaining to the coefficients of the particular solutions in the distant field include elements adjacent to the far boundary of the computed field. For the term representing the circulation the support region in the ϕ,ψ representation is given by the entire flow field. In the $\vec{w}$ representation J is given by Eq. (115) that is

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$$J = (1/2) \int_{\Omega} [\rho_0^{-2}(\text{div } \vec{m}(\vec{w}))^2 + (D \text{ curl } \vec{w})^2] d\tau \qquad (249)$$

For the ϕ, ψ representation one has from Eq. (142) combined with Eq. (151)

$$J = (1/2) \int_{\Omega} [D(\vec{m}(\text{grad } \phi) - \vec{e}_x \psi_y + \vec{e}_y \psi_x)^2] d\tau \qquad (250)$$

For simplicity additional terms needed because of special requirements, for instance the presence of shocks, or approximations to the boundary conditions are omitted. In these representations it is probably preferable to exclude immediately from $\vec{w}$, ϕ and ψ the contributions of the underlying parallel flow.

For some point of the respective function spaces, (given by the shape parameters including the coefficients of the particular solutions in the distant field) the function $\vec{w}$ or ϕ and ψ are completely determined and the integral J can be evaluated. The derivatives $\partial J/\partial p_j$ and $\partial J/\partial c_j$ (where the p_j's and the c_j's are respectively the shape parameters for the interior of the computed flow field and the coefficients of the particular solutions in the distant field) are best found by numerical differentiation of J, although it is possible to derive integral representations.

Dominant at infinity are the contributions due to a circulation. On the other hand, we have found that there is a linear combination of the contribution due to the circulation and a function without circulation which leave the integral J unchanged. If one dealt with linear equations (as in the incompressible case) one could determine such a linear combination once and for all. For the present nonlinear case this is impossible, but it seems desirable to modify the formulation in such a manner that large contributions from distant parts of the flow field do not appear. Since ψ is single valued it is in principle not necessary, except at the far boundary, to connect variations of the circulation to variations of ψ, but by

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doing so one obtains an integrand which in distant parts of the flow field is much smaller. Let $c_\Gamma \phi^{(\Gamma)}$ be the contribution to ϕ due to circulation. Let $c_\Gamma \psi^{(\Gamma)}$ be the corresponding contribution to ψ and

$$\psi = \psi' + c_\Gamma \psi^{(\Gamma)}$$

$$\phi = \phi' + c_\Gamma \phi^{(\Gamma)}$$

Then one has

$$J = (1/2) \int_\Omega [D(\vec{m}(\text{grad } (\phi' + c_\Gamma \phi^{(\Gamma)}) - \vec{e}_x \frac{\partial}{\partial y}(\psi' + c_\Gamma \psi^{(\Gamma)})$$

$$+ \vec{e}_y \frac{\partial}{\partial x}(\psi' + c_\Gamma \psi^{(\Gamma)})]^2 d\tau$$

Actually nothing except the description of the elements of the function space by ϕ', ψ', $\phi^{(\Gamma)}$, and $\psi^{(\Gamma)}$ has been changed. But the functions ϕ' and ψ' vanish more rapidly as one moves away from the profile, and at distant parts of the computed flow field the contribution of $\phi^{(\Gamma)}$ and $\psi^{(\Gamma)}$ cancels the expression $\vec{m}$ almost completely. A similar procedure may be practical also in the $\vec{w}$ representation. In principle the circulation enters the $\vec{w}$ representation only through data at the far boundary. One can, however, define within the computed part of the flow field a portion $c_\Gamma \vec{w}^{(\Gamma)}$ in such a manner that $\vec{w}^{(\Gamma)}$ satisfies

$$\text{div}(\rho_0 \beta^2 D^{-2} \vec{w}^{(\Gamma)}) = 0$$

$$\text{curl}(\vec{w}^{(\Gamma)}) \qquad = 0$$

This means one extends the region of support of the function $\vec{w}^{(\Gamma)}$ which pertains to the contribution of the circulation in the distant field throughout the flow field, rather than just through the elements adjacent to the far boundary. Then one writes

$$\vec{w}^{(\Gamma)} = \vec{w}' + c_{(\Gamma)} \vec{w}^{(\Gamma)}$$

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If one uses this form $\vec{w}'$, it will decrease more quickly and the terms of the integrand will cancel almost completely at a distance from the air foil. This holds in particular when one determines $\partial J/\partial c_r$. Of course, the computational work for determining $\partial J/\partial c_r$ is increased because the support of $\vec{w}^{(r)}$ is given by the whole flow field.

The distance definition required for a conjugate gradient search is given in the $\vec{w}$ representation by Eq. (117)

$$[\vec{g}, \delta\vec{w}] = \sum_{i=1}^{2} \int_{\Omega+\Omega_D} (D^{-2}\text{grad } \vec{g}_i \cdot \text{grad } \delta\vec{w}_i)d\tau \qquad (251)$$

(Subscripts 1 and 2 refere as always to the x and y components of the vectors $\vec{g}$ and $\vec{w}$. We have chosen $c_5 = 0$ in Eq. (117).) In the distant field one has the representation

$$\vec{w} = \sum_{\ell} c_{\ell} \vec{w}^{(\ell)}$$

and

$$\vec{g} = \sum_{n} c_{g,n} \vec{w}^{(n)}$$

where $\vec{w}^{(\ell)}$ refer to particular solutions admitted in the distant field, one of the particular solutions is $\vec{w}^{(r)}$ and the corresponding c_{ℓ} is the coefficient c_r. They are of course the same for $\vec{w}$ and $\vec{g}$. Each component satisfies

$$\text{div}(D^{-2}\text{grad } \vec{w}_i^{(\ell)}) = 0$$
$$\text{div}(D^{-2}\text{grad } \vec{w}^{(r)}) = 0 \qquad (252)$$

To be varied are the c_{ℓ} and c_r. Therefore, one has in the distant field

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$$\sum_{i=1}^{2} \int_{\Omega_D} (D^{-2} \mathrm{grad}\, g_i) \cdot (\mathrm{grad}\, \delta w_i) d\tau$$

$$= \sum_{i=1}^{2} \sum_{\ell n} c_{g,n} \delta c_\ell \int_{\Omega_D} (D^{-2} \mathrm{grad}\, g_i^{(n)} \cdot \mathrm{grad}\, w_i^{(\ell)}) d\tau$$

$$= \sum_{i=1}^{2} \sum_{\ell n} c_{g,n} \delta c_\ell \int_{\Gamma_D} (D^{-2} \mathrm{grad}\, g_i^{(n)} \cdot \vec{e}_{d,n}) w_i^{(\ell)} do$$

The last step is obtained by an integration by parts. The integral over Ω_D vanishes because of Eqs. (252), $\vec{e}_{d,n}$ denotes the unit vector in the direction of the outer normal to the distant field. If one denotes simply by $\vec{e}_n$ the outer normal of the far boundary of the computed flow field then one has

$$\vec{e}_n = -\vec{e}_{d,n}$$

Thus from Eq. (251)

$$[\vec{g}, \delta\vec{w}] = \sum_{i=1}^{2} \left(\int_\Omega D^{-2} \mathrm{grad}\, g_i \cdot \mathrm{grad}\, \delta\vec{w}_i \right) d\tau$$

$$- \sum_{\ell n} \delta c_\ell c_{g,n} \int_{\Gamma_D} (D^{-2} \mathrm{grad}\, \vec{w}_i^{(n)} \cdot \vec{e}_{f,n}) \vec{w}_i^{(\ell)} do) \tag{253}$$

This expression is equated with δJ, expressed in terms of $\delta\vec{w}$ and δc_ℓ. The variations $\delta\vec{w}$ are, of course, expressed by variations of the shape parameters. For each variation of a shape parameter and of c_ℓ one obtains one linear equation for the coefficients $c_{g,i}$ and the shape parameters pertaining to $\vec{g}$ ($q_{i,j}$ in Eq. (84)).

In the ϕ, ψ representation one defines according to Eq. (172)

$$[(g_\phi, g_\psi), (\delta\phi, \delta\psi)] = \int\limits_{\Omega + \Omega_D} [(D^{-2} \text{grad } g_\phi \cdot \text{grad } \delta_\phi$$

$$+ D^{-2} \text{grad } g_\psi \cdot \text{grad } \delta_\psi)] d\tau \qquad (254)$$

Here we have used the fact that for the two-dimensional problem

$$A_1 = 0, \; A_2 = 0, \; A_3 = \psi.$$

In the presence of circulation one encounters the following difficulty in the definition of the scalar product between two functions $\phi^{(1)}$ and $\phi^{(2)}$. These functions will contain terms $c_{r1} \text{arctg}(\beta y/x)$ and $c_{r2} \text{arctg}(\beta y/x)$, respectively. The corresponding functions ψ are $-(c_{r1}/2)\rho_0 \log(x^2 + \beta^2 y^2)$ and $-c_{r2}\rho_0 \log(x^2 + \beta^2 y^2)$. This gives rise to expressions in the scalar product

$$c_{r1}c_{r2} (D^{-1}(\text{grad arctg}(\beta y/x))^2 d\tau$$

Now $d\tau \sim rdrd\theta$, grad $\text{arctg}(\beta y/x) \sim r^{-1}$. The integral does not converge as r tends to infinity.

This can be remedied by redefining the scalar product. Two such formulations are given

1. The values of c_r can always be recognized in an element of the functions space by examining the jump of ϕ as one makes one complete circuit around the profile. The functions $\phi^{(\Gamma)}$ and $\psi^{(\Gamma)}$ in the distant field have already been defined. We add a definition for these functions within the computed part of the flow field. The only requirements are that $\phi^{(\Gamma)}$ gives the same jump of ϕ in the computed part of the flow field and continuity as one passes from the computed part of the flow field to the distant field. No relation to the first differential equations need to be imposed. $\psi^{(\Gamma)}$ has no jump, otherwise one has analogous conditions. One now represents an element (ϕ, ψ) as

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$$(\phi', \psi') + c_\Gamma(\phi^{(\Gamma)}, \psi^{(\Gamma)}) \tag{255}$$

and uses as definitions of the scalar product

$$[(\phi^{(1)}, \psi^{(1)}), (\phi^{(2)}, \psi^{(2)})]$$

$$= \int_{\Omega + \Omega_d} (D^{-1} \text{grad } \phi'^{(1)}) \cdot (D^{-1} \text{grad } \phi'^{(2)}) d\tau \tag{256}$$

$$+ \int_{\Omega + \Omega_d} (D^{-1} \text{grad } \psi'^{(1)}) \cdot (D^{-1} \text{grad } \psi'^{(2)}) d\tau + \text{const } c_{\Gamma 1} c_{\Gamma 2}$$

2. One can introduce a contour Γ_c (which can but need not coincide with the outer contour of the computed flow field Γ_D). Outside of Γ_c one disregards in the definition of the scalar product the terms due to circulation. Let Ω_i and Ω_o be the fields inside and outside of Γ_c. Accordingly we define

$$[(g_\phi, g_\psi), (\delta\phi, \delta\psi)] = \int_{\Omega_i} [D^{-2} \text{grad } g_\phi \cdot \text{grad } \delta\phi + D^{-2} \text{grad } g_\psi \cdot \text{grad } \delta\psi] \delta\tau$$

$$\tag{257}$$

$$+ \int_{\Omega_o} [D^{-2} \text{grad}(g_\phi - c_g^{(\Gamma)} \phi^{(\Gamma)}) \cdot \text{grad}(\delta\phi - \delta c_g^{(\Gamma)} \phi^{(\Gamma)})$$

$$+ D^{-2}(\text{grad } g_\psi - c_g^{(\Gamma)} \psi^{(\Gamma)}) \cdot \text{grad}(\delta\psi - \delta c_g^{(\Gamma)} \psi^{(\Gamma)}] d\tau$$

This leads again to a legitimate positive definite distance definition. The same difficulty will be encountered in the ζ representation. One then obtains the following expressions. Assume for the second approach that

$$\Omega_D = \Omega_c$$

In the distant field one then has in each case

$$\phi' = \sum_{\ell} c_{\ell} \phi^{(\ell)}$$

$$\psi' = \sum_{\ell} c_{\ell} \psi^{(\ell)}$$

$$g'_{\phi} = \sum_{n} c_{g,n} \phi^{(n)}$$

$$g'_{\psi} = \sum_{n} c_{g,n} \psi^{(n)}$$

(258)

Expressions for the particular solution $\phi^{(\ell)}$ and $\psi^{(\ell)}$ have been derived in the preceding section. In the distant field expressions for the circulation are not included. The function $\phi^{(\ell)}$ and $\psi^{(\ell)}$ satisfy

$$\mathrm{div}(D^{-2}\mathrm{grad}\ \phi^{(\ell)}) = 0$$

$$\mathrm{div}(D^{-2}\mathrm{grad}\ \psi^{(\ell)}) = 0$$

(259)

One then obtains for the distant field

$$\int_{\Omega_D} [D^{-2}\mathrm{grad}\ g_{\phi}\cdot\mathrm{grad}\ \delta\phi]d\tau = \sum_{\ell n} c_{g,n}\delta c_{\ell}\int_{\Omega_d} (D^{-2}\mathrm{grad}\ \phi^{(\ell)}\cdot\ \mathrm{grad}\ \phi^{(n)})d\tau$$

Because of Eq. (259) and Eq. (36) one has

$$\int_{\Omega_D} D^{-2}\mathrm{grad}\ \phi^{(\ell)}\cdot\ \mathrm{grad}\ \phi^{(n)}d\tau = \int_{\Gamma_D} ((D^{-2}\mathrm{grad}\ \phi^{(\ell)})\cdot\ \vec{e}_{dn})\phi^{(n)}do$$

$$= \int_{\Gamma_{\Gamma}} ((D^{-2}\mathrm{grad}\ \phi^{(n)})\cdot\ \vec{e}_{dn})\phi^{(\ell)}do$$

(260)

Analogous formulae are obtained for ψ.

In the first approach one uses the decomposition Eq. (255) throughout the flow field, ϕ' and ψ', and g'_{ϕ}, g'_{ψ} are represented in the computed part of the flow field by means of elemental shape functions. One then has according to Eq. (256)

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$$[(g_\phi, g_\psi, c_g^{(\Gamma)}), (\delta\phi, \delta\psi, \delta c^{(\Gamma)})]$$

$$= \int_\Omega [(D^{-2} \mathrm{grad}\, g'_\phi) \cdot \mathrm{grad}\, \delta\phi' + (D^{-2} \mathrm{grad}\, g'_\psi) \cdot \mathrm{grad}\, \delta\psi'] d\tau$$

$$- \underset{\ell n}{\Sigma\Sigma} c_{g,n} \delta c_\ell \int_{\Gamma_D} [(D^{-2} \mathrm{grad}\, \phi^{(\ell)}) \cdot \vec{e}_n \phi^{(n)} + (D^{-2} \mathrm{grad}\, \psi^{(\ell)} \cdot \vec{e}_n) \psi^{(n)}] do$$

$$+ \mathrm{const}\, c_g^{(\Gamma)}\, \delta c_g^{(\Gamma)}$$

As before, the shape function belonging to the variations expressed by δc_ℓ include in their support area elements adjacent to the distant boundary of the flow field.

In the second approach the integrals over the computed part of the flow field include the contributions of $\phi^{(\Gamma)}$ and $\psi^{(\Gamma)}$ to ϕ, g_ϕ and ψ, g_ψ. These expressions will, therefore, depend upon $\delta c_g^{(\Gamma)}$ and $c_g^{(\Gamma)}$. (At least in ϕ and g_ϕ one must identify in the finite element representation a contribution due to circulation, because of the jump in ϕ after one circuit). One then obtains

$$\delta J = [(g_\phi, g_\psi, c_g^{(\Gamma)}), (\delta\phi, \delta\psi, \delta c_g^{(\Gamma)})]$$

$$= \int_\Omega [(D^{-2} \mathrm{grad}\, g_\phi) \cdot \mathrm{grad}\, \delta\phi + (D^{-2} \mathrm{grad}\, g_\psi) \cdot \mathrm{grad}\, \delta\psi] d\tau$$

$$- \underset{\ell n}{\Sigma\Sigma} c_{g,n} \delta c_\ell \left(\int_{\Gamma_D} [D^{-2} \mathrm{grad}\, \phi^{(\ell)} \cdot \vec{e}_n) \phi^{(n)} + (D^{-2} \mathrm{grad}\, \psi^{(\ell)} \cdot \vec{e}_n) \psi^{(n)}] do \right.$$

To obtain equations for g_ϕ and g_ψ, one equates these expressions (in terms of $g_\phi, g_\psi, \delta\phi,$ and $\delta\psi$) with those in terms of $\phi, \psi, \delta\phi,$ and $\delta\psi$.

The reader will keep in mind that functions ϕ and ψ, and also g_ϕ and g_ψ include terms for the circulation throughout the flow field. Merely in the expression for the distance definition is the contribution of the circulation omitted in the field Ω_D. No infinities of the type encountered in the distance definition

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(which are the cause of the above revisions) arise in the expressions δJ in terms of ϕ, ψ, $\delta\phi$, and $\delta\psi$, for these expressions are ultimately based on the flow equations which are satisfied in the distant field.

In the Bristeau approach an auxiliary function ζ is determined by Eq. (176)

$$\mathrm{div}(D^{-2}\mathrm{grad}\ \zeta) = \mathrm{div}(\vec{m}(\mathrm{grad}\ \phi))$$

with boundary conditions Eqs. (180) and (181)

$$\zeta = 0 \quad \text{on}\ \Gamma - \Gamma_2$$
$$D^{-2}\mathrm{grad}\ \zeta \cdot \vec{e}_n - (m(\vec{\mathrm{grad}}\ \phi) \cdot \vec{e}_n - f_2) = 0 \quad \text{on}\ \Gamma_2$$

The function ζ provides a measure for the residual $\mathrm{div}(\vec{m}(\mathrm{grad}\ \phi))$; the differential equation for ζ is not related in its form to the original differential equation. (The equation for ζ is a Poisson equation, which is elliptic, while the technique can be applied also to hyperbolic equations.) The function ζ obtained from the last equation is single valued in the first place. In the flow over a profile one might, however, consider adding a term corresponding the circulation in order to suppress the (fairly harmless) singularity which otherwise would arise at the trailing edge. The author believes that this possibility should be rejected because of the following considerations.

1. The function J defined by means of ζ

$$J = \int_{\Omega + \Omega_D} (\mathrm{grad}\ \zeta)^2 d\tau$$

would not converge in a finite domain.

2. An even more compelling reason arises already for a finite region in the evaluations of δJ. Consider such a region Ω. If a term analogous to a circulation is present then ζ will have a jump and one must introduce a cut which connects the

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profile with the outer boundary. Retracing previous steps one obtains

$$\delta J = \int_{\Omega} D^{-2} \text{grad } \xi \cdot \text{grad } \delta\xi d\tau$$

$$= \int_{\Gamma} \xi(\text{grad } D^{-2}\delta\xi \cdot \vec{e}_n)do - \int_{\Omega} \xi \text{div } D^{-2}\text{grad } \delta\phi$$

but

$$\text{div}(D^{-2}\text{grad } \delta\xi) = \text{div}(M \text{ grad } \delta\phi)$$

Therefore, with another integration by part

$$\delta J = \int_{\Gamma} \xi(\text{grad}(D^{-2}\delta\xi - M \text{ grad } \delta\phi) \cdot \vec{e}_n do$$

$$+ \int_{\Omega} M \text{ grad } \xi \cdot \text{grad } \delta\phi d\tau$$

For reasons given above the integral over Γ vanishes for the surface of the profile and for the outer boundary, but if ξ has a jump $\xi|_-^+$ a contribution from the two borders of the cut remains. It is given by

$$\xi|_-^+ \int_{\Gamma_{cut}} (\text{grad } \delta\xi - M \text{ grad } \delta\phi) \cdot \vec{e}_n do$$

In evaluating this expression one is confronted with the forbidding task of determining for each choice of $\delta\phi$ the pertinent $\delta\xi$. A term in ξ analogous to a circulation will therefore be excluded. Conceptually (but not practically) one can express the contribution to ϕ due to a change of the circulation by a function $\phi^{(\Gamma)}$ which satisfies the differential equation linearized for the current approximation, namely,

$$\text{div}(M \text{ grad } \phi^{(\Gamma)}) = 0$$

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and the homogeneous boundary conditions at the profile. The function $\xi^{(\Gamma)}$ pertaining to this function $\phi^{(\Gamma)}$ is zero, because the right-hand side of the equation determining ξ, viz.

$$\text{div}(D^{-2}\text{grad }\xi^{(\Gamma)}) = \text{div}(M \text{ grad } \phi^{(\Gamma)})$$

is zero. One therefore has

$$J + \delta J = \int_{\Omega} [\text{grad } \xi + \delta\Gamma \text{ grad } \xi_\Gamma + 0(\delta\Gamma)^2]^2 d\tau$$

$$= J + 2\delta\Gamma \int_{\Omega} [\text{grad } \xi \cdot \text{grad } \xi_\Gamma + 0(\delta\Gamma)^2] d\tau$$

$$\delta J = 0(\delta\Gamma)^2 \qquad\qquad (262)$$

This result expresses the fact, stated before, that in the ϕ space the condition $J = 0$ does not define a unique point but there is a "valley" along which $J = 0$. The existence of such a valley remains, although Eq. (262) will then not be satisfied, if the function $\phi^{(\Gamma)}$ fails to satisfy Eq. (261). In the future development it is <u>not</u> assumed that Eq. (261) is satisfied by $\phi^{(\Gamma)}$, except for the distant field where we postulate that the approximation to Eq. (261)

$$\text{div}(D^{-2}\text{grad }\phi^{(\Gamma)}) = 0$$

is satisfied.

The discretized form of the equation for ξ is conveniently obtained from a minimum formulation. This is a technicality which has nothing to do with residual minimization applied to the basic flow equations. Such equations are obtained by minimizing

$$\int_{\Omega} [(D\vec{m}(\text{grad } \phi) - D^{-1}\text{grad }\xi]^2 d\tau + \int_{\Gamma_2} f_2\xi \, do$$

with $\xi = 0$ on $(\Gamma - \Gamma_2)$.

The function ξ is represented by the same shape functions as those used for the function ϕ, including those in the field at infinity. There ϕ is supposed to satisfy the linearized form of

$$\text{div } \vec{m}(\text{grad } \phi) = \beta^2 \text{div } D^{-2}\text{grad } \phi = 0.$$

In the distant field, contributions due to circulation are omitted. The variation of J is given according to Eq. (183) by

$$\delta J = \int_{\Omega+\Omega_d} (M \text{ grad } \xi \cdot \text{ grad } \delta\phi)d\tau$$

ϕ consists of a single-valued part ϕ' and a portion due to the circulation

$$\phi = \phi' + c_\Gamma \phi^{(\Gamma)} .$$

Then

$$\delta\phi = \delta\phi' + \delta c_\Gamma \phi^{(\Gamma)}$$

Along some line, which extends from the profile to infinity, the function $\phi^{(\Gamma)}$ has a constant jump. The integral will converge because ξ is assumed to be single valued, and because in the distant field ϕ is assumed to satisfy the flow equation linearized for the vicinity of a parallel flow. Specifically, one has in the distant field

$$\phi = \sum_\ell c_\ell \phi^{(\ell)} + c_\Gamma \phi^{(\Gamma)}$$

and

$$\xi = \sum_n c_\xi^{(n)} \phi^{(n)}$$

where as before the functions $\phi^{(\ell)}$ are those single-valued particular solutions of the linearized flow equation which are

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admitted, and $\phi^{(\Gamma)}$ is the one which expresses a circulation. Then one has as contribution to δJ from the distant field

$$\int_{\Omega_D} (\rho_0 \beta^2 D^{-2} \text{grad } \xi \cdot \text{grad } \delta\phi) d\tau$$

$$= \rho_0 \beta^2 \sum_{\ell} \delta c_\ell \int (D^{-2} \text{ grad } \phi^{(\ell)}) \cdot \text{grad } \xi) d\tau$$

$$+ \delta c_\Gamma \int_{\Omega_D} (D^{-2} \text{ grad } \phi^{(\Gamma)}) \cdot \text{grad } \xi \, d\tau$$

$$= \rho_0 \beta^2 \sum_{\ell} \delta c_\ell \int_{\Gamma_D} (D^{-2} \text{grad } \phi^{(\ell)} \cdot \vec{e}_{n,d}) \xi \, do$$

$$+ \delta c_\Gamma \int_{\Gamma_D} (D^{-2} \text{grad } \phi^{(\Gamma)} \cdot \vec{e}_{n,d}) \xi \, do$$

(The contour Γ_D should include a circle in the x,y system which moves to infinity). This contribution vanishes because $D^{-2} \text{grad } \phi^{(\Gamma)} \cdot \vec{e}_{n,d}$ vanishes. The integrals over Ω_D vanish because $\phi^{(\ell)}$ and $\phi^{(\Gamma)}$ satisfy the linearized flow equations. One therefore has

$$\delta J = \int_{\Omega} (M \text{ grad } \xi \cdot \text{grad } \delta\phi) d\tau - \sum_{\ell} \delta c_\ell \rho_0 \beta^2 \int_{\Gamma_D} (D^{-2} \text{grad } \phi^{(\ell)} \cdot \vec{e}_{n,d}) \xi \, do$$

$$- \delta c_\Gamma \rho_0 \beta^2 \int_{\Gamma_D} (D^{-2} \text{grad } \phi^{(\Gamma)} \cdot \vec{e}_{n,d}) \xi \, do$$

(263)

In the discretized case the shape functions belonging to the shape parameters c and c_Γ include in their support respectively the elements adjacent to the far boundary of the computed flow field or the entire flow field.

In the following discussion of tne gradient vector one will remember that the distance definition in the ϕ space is conceptually independent from the definition of J. In an infinite space, the Bristeau scalar product

$$[\phi^{(1)}, \phi^{(2)}] = \int\limits_{\Omega + \Omega_D} D^{-2} \text{grad } \phi^{(1)} \cdot \text{grad } \phi^{(2)} d\tau$$

is infinite if both $\phi^{(1)}$ and $\phi^{(2)}$ contain a term due to circulation, but such terms must be admitted in the gradient vector as well as in the potential. In complete analogy with the ϕ, ψ representation write

$$\phi^{(1)} = \phi'^{(1)} + c_{\Gamma 1} \phi^{(\Gamma)}$$
$$\phi^{(2)} = \phi'^{(2)} + c_{\Gamma 2} \phi^{(\Gamma)}$$

(264)

where $\phi'^{(1)}$ and $\phi'^{(2)}$ are single valued and satisfy in Ω_d

$$\text{div } D^{-2} \text{grad } \phi' = 0$$

$\phi^{(\Gamma)}$ has a constant jump and satisfies tne corresponding differential equation in the distant field but not necessarily in tne computed part of the flow field. Then we define either

$$[\phi^{(1)}, \phi^{(2)}] = \int\limits_{\Omega + \Omega_d} (D^{-1}(\text{grad } \phi'^{(1)}) \cdot (D^{-1} \text{grad } \phi'^{(2)}) d\tau$$

(265)

$$+ \text{ const } c_{\Gamma 1} \cdot c_{\Gamma 2}$$

or alternatively

$$[\phi^{(1)}, \phi^{(2)}] = \int\limits_{\Omega_i} (D^{-1} \text{grad } \phi^{(1)}) \cdot (D^{-1} \text{grad } \phi^{(2)}) d\tau$$

$$+ \int\limits_{\Omega_o} (D^{-1} \text{grad } \phi'^{(1)}) \cdot (D^{-1} \text{grad } \phi'^{(2)}) d\tau$$

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Both definitions give positive definite norms which do not
diverge if circulation is present.

Now we choose for $\phi^{(1)}$ the gradient function g and for $\phi^{(2)}$
the variation $\delta\phi$. We set in analogy to Eq. (264)

$$g = g' + c_g^{(\Gamma)}\phi^{(\Gamma)}$$

$$\delta\phi = \delta\phi' + \delta c_\Gamma \phi^{(\Gamma)}$$

Then, for the definition Eq. (265)

$$\delta J = [g,\delta\phi] = \int_{\Omega+\Omega_D} (D^{-1}grad\ g') \cdot (D^{-1}grad\ \delta\phi')d\tau$$

$$+ \ const\ c_g^{(\Gamma)}\delta c_{(\Gamma)}$$

In the distant field one has

$$g' = \sum_n c_n^{(\Gamma)}\phi^{(n)}$$

$$\delta\phi' = \sum_\ell \delta c_\ell \phi^{(\ell)}$$

In a familiar manner the integral over the distant field can be
changed into a surface integral.

$$\int_{\Omega_D} (D^{-2}grad\ g')\cdot grad\ \delta\phi' d\tau = \sum_{n\ell} c_g^{(\ell)}\delta c_\ell \int_{\Gamma_D} (D^{-2}grad\ \phi^{(n)}\cdot\vec{e}_{n,d})\phi^{(\ell)}do$$

$$= \sum_{n\ell} c_g^{(\ell)}\delta c_\ell \int_\Gamma (D^{-2}grad\ \phi^{(\ell)}\cdot\vec{e}_{n,d})\phi^{(n)}do.$$

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Thus, for the first definition

$$\delta J = \int_{\Omega} (D^{-1} \text{grad } g') \cdot (D^{-1} \text{grad } \delta\phi') d\tau$$

(266)

$$- \sum_n \sum_\ell c_g^{(\ell)} \delta c_\ell \int_{\Gamma_D} (D^{-2} \text{grad } \phi^{(\ell)} \cdot \vec{e}_n) \phi^{(n)} do + \text{const } c_g^{(\Gamma)} \delta c_\Gamma$$

The support regions belonging to the shape function whose parameters are $c_g^{(\ell)}$ and δc_ℓ include, as before, the elements adjacent to the boundary Γ_D. The gradient vector is then obtained by equating the expression Eq. (263) and Eq. (266) and using the fact that the parameter pertaining to $\delta\phi$ (including δc_Γ and δc_ℓ) can be varied independently

A similar procedure can be applied to the second definition of the scalar product. Details are left to the reader. It is difficult to decide on purely theoretical grounds which of these two definitions is preferable.

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SECTION X
THREE-DIMENSIONAL FLOWS WITH A WAKE

In formulations of the problem which use the potential (including those which use the potential and the vector field $\vec{A}$) one must introduce instead of the wake an internal boundary. This can be done also in the $\vec{w}$ representation. The procedure is described in principle in Section VIII. Outside of the wake one then has

$$\text{curl } \vec{w} = 0$$

$$\text{div } \vec{m}(\vec{w}) = 0 \qquad (267)$$

or
$$\vec{w} = \text{grad } \phi$$

$$\text{div } \vec{m}(\text{grad } \phi) = 0 \qquad (268)$$

or
$$\vec{w} = \text{grad } \phi$$

$$\vec{m}(\text{grad } \phi) = \text{curl } \vec{A} \qquad (269)$$

We have seen in Section VII, that the ability of an Euler code to capture a wake is preserved, if one uses instead of Eq. (267)

$$\vec{w} \times \text{curl } \vec{w} = 0$$

$$\text{div } \vec{m}(\vec{w}) = 0 \qquad (270)$$

Since the wake is not explicitly defined in such an approach (in particular not the line from which the wake starts) one has in addition the conditions, imposed along the entire trailing edge, that the velocity immediately behind the trailing edge has the direction of the bisector of the angle at the trailing edge. This is quite analogous to the treatment of the two-dimensional case, where one does not introduce the wake explicitly.

Applying residual minimization one defines

$$J = \int_\Omega [(\text{div } \vec{m}(\vec{w}))^2 + \text{const}(\vec{w} \times \text{curl } \vec{w})^2 d\tau \qquad (271)$$

(with an additional condition at the trailing edge). This then should allow wake capturing. Strictly speaking such a formulation has a meaning only in the discretized form. At the wake, $\vec{w}$ is discontinuous, the vorticity curl $\vec{w}$ is zero everywhere except within the infinitesimally thin wake, but there it has the character of a delta function. In the nondiscretized form $\int (\vec{w} \times \text{curl } \vec{w})^2 d\tau$ has no meaning, even if one applies the concept of generalized functions in the evaluation of the integral. The discretized version camouflages this difficulty, although it may make itself felt, if one reduces the size of the elements.

Frequently the vortex sheet is treated in the linearized approximation. Then $\vec{w} \times \text{curl } \vec{w}$ is replaced by $\vec{e}_x \times \text{curl } \vec{w}$ and since

$$\text{div curl } \vec{w} = 0$$

one obtains

$$\text{curl } \vec{w} = f(y,z) \cdot \vec{e}_x.$$

The vorticity is constant along the streamlines of the linearized flow, and this implies that the jump of the potential between the upper and lower side of the wake is constant along those streamlines. Of course the magnitude of the jumps is unknown, it is represented by a function of one independent variable (for instance z). The solution of the linearized problem can then be obtained by determining the velocity field due to a doublet distribution (with axes directed normal to the vortex sheet) whose intensity is determined by the local jump of the potential. To this expression is superimposed a solution of the homogeneous problem (which disregards the wake), chosen in such a manner that

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the boundary condition at the surface of the wing is satisfied. The velocity field obtained by this construction will, in general, fail to satisfy the Kutta condition at the trailing edge, this condition requires the proper choice of the potential jumps in the wake.

Incidentally, in the nonlinearized formulation the form of the flow field in the vicinity of a trailing edge is by no means simple. The velocity components on the upper and lower side in the direction of the trailing edge are in general not the same. Let us assume that one has a finite trailing edge angle. If the flow behind the trailing edge has the direction of the bisector of this angle, then one has for the component normal to the trailing edge a stagnation point on both sides. But if the tangential velocities are different, then the "stagnation pressures" are different. Therefore, this description (which we have used above) is not entirely correct. The author conjectures that on the side where the tangential velocity is smaller, the streamlines do not reach a stagnation point, but only a velocity which gives the pressure corresponding to the stagnation pressure (for the normal flow component) on the other side. The streamline retains the direction of the surface (but probably with infinite curvature). On the other side one has a sudden change of direction. Of course these details will not appear in the discretized treatment of the problem, and then the rule that the velocity vector should have the direction of the bisector is sufficient.

In one approach to the treatment of the computed part of the flow field one identifies the wake. In other words one introduces parameters describing the wake shape by a finite number of parameters. The characterization of the element in the function space then includes the shape of the vortex sheet. The definition of the integral to be minimized and the definition of the scalar product must include the parameters for the shape of the vortex sheet. Alternatively one does not identify the wake

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and uses the formulation (Eq. (271)) (with extra terms which express the Kutta condition).

Also needed are far field conditions. The most important term is the expression due to the vortex sheet. For the flow field in the vicinity of the vortex sheet the application of linearized theory is somewhat questionable; for linearized theory will give infinite velocities at the outer edges of the vortex sheet. The results of the linearized theory will nevertheless be used because one needs analytic solutions for the distant field and nothing better is available. The far field affects the computed part of the flow field only through the velocity distribution at the far boundary, but at best this velocity distribution is only approximately known.

The vortices in the far field are represented by straight lines parallel to the x-axis. They may or may not lie in a plane. The parameters describing the vortex sheet give in one form or the other the geometric location of these vortices and their strength. As was mentioned above, only the velocity distribution at the boundary of the computed region due to such a vortex sheet is needed. It is practical to terminate the vortex sheet inside of the computed region (for instance at the wing itself). An extension of the vortex sheet into the computed field adds solutions of the homogeneous problem to the outer field and this is always admissible. (Inside of the computed field the vortex sheet in the far field is not used.) By such an extension one avoids singularities which arise at the beginning of the vortex sheet, in this case at the downstream surface of the computed field. (Singularities which arise at the edges of the vortex sheet are, of course, unavoidable.) In principle one can compute the flow field by a superposition of panels of dipoles of constant intensities. This amounts to building up the vortex sheet by means of a number of horseshoe vortices. At the panel boundaries one then would obtain infinite velocities (with opposite sign normal to the vortex sheet). This can be avoided if one represents the vortex sheet by one field with vortices which extend from upstream to downstream infinity and subtracts

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from it the same vortex sheet extending from upstream infinity to some line that crosses the wake close to the wing. The velocity pertaining to the first vortex sheet can be expressed by two-dimensional methods. For the boundary of the computed field downstream of the wing, the second vortex field can be taken into account by a fairly rough approximation (for instance horseshoe vortices). For the boundary of the computed region upstream of the wing one uses the original form of the vortex sheet in a fairly rough approximation. (This however is a technicality.) To the expressions so obtained one may add in the distant field the first few of the particular solutions derived in Section VIII.

The flow field as a whole is now described by the shape parameters for the interior of the field within the computed field (they include the location of the vortex sheet, if one introduces it separately) and the parameters which characterize the distant field: in the first place the parameters for the wake, and perhaps the first few of the homogeneous particular solution for the outer field given in Section VIII.

If one treats far field conditions in this manner, then some mismatch between the vortex sheet in the computed part of the flow field and in the distant field is unavoidable. In the method of wake capturing sketched above the vortex sheet appears in the interior as a region (at least of the width of the mesh) of vorticity while in the far field one is likely to employ a localized vortex sheet. The following simple example is shown in order to snow the effects of such a mismatch.

We consider the linearized problem. A plane x = const takes the place of a smooth body on whose surface a line is marked at which vortices are allowed to emerge. The far boundary is represented by a plane x = L. We study particular solutions of the Euler equations for the region 0 < x < L. For simplicity an incompressible flow is considered. Then one has

$$\text{div } \vec{w} = 0$$

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or

$$\partial w_1/\partial x + \partial w_2/\partial y + \partial w_3/\partial z = 0 \qquad (272)$$

The condition

$$\vec{w} \times \text{curl } \vec{w} = 0$$

in linearized approximation gives

$$\vec{e}_x \times \text{curl } \vec{w} = 0$$

or, written in detail

$$\partial w_1/\partial z - \partial w_3/\partial x = 0$$
$$\qquad (273)$$
$$\partial w_1/\partial y - \partial w_2/\partial x = 0$$

One defines, accordingly

$$J = (1/2)\int \{(\partial w_1/\partial x + \partial w_2/\partial y + \partial w_3/\partial z)^2$$
$$\qquad (274)$$
$$+ + (\partial w_1/\partial z - \partial w_3/\partial x)^2 + (\partial w_1/\partial y - \partial w_2/\partial x)^2 dxdydz = \text{Min}$$

Independent variations of δw_1, δw_2, and δw_3 lead to the Euler equations

$$\Delta w_1 = 0 \qquad (275)$$

$$\partial^2 w_2/\partial x^2 + \partial^2 w_2/\partial y^2 + \partial^2 w_3/(\partial y \partial z) = 0$$
$$\qquad (276)$$
$$\partial^2 w_3/\partial x^2 + \partial^2 w_2/(\partial y \partial z) + \partial^2 w_3/\partial z^2 = 0$$

These equations are, of course, satisfied if one has a potential flow

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$$w_1 = \phi_x, \quad w_2 = \phi_y, \quad w_3 = \phi_z$$

provided that

$$\Delta \phi = 0.$$

Eqs. (275) and (276) are independent of each other. Particular solutions of Eqs. (276) are obtained by the hypothesis

$$w_2(x,y,z) = \tilde{w}_2 \exp(\lambda x + i(\beta y + \gamma z))$$

$$(277)$$

$$w_3(x,y,z) = \tilde{w}_3 \exp(\lambda x + i(\beta y + \gamma z))$$

where $\tilde{w}_2$ and $\tilde{w}_3$ are constants. Then, by substitution in Eqs. (275)

$$(\gamma^2 - \beta^2)\tilde{w}_2 - \beta\gamma\,\tilde{w}_3 = 0$$

$$-\beta\gamma\,\tilde{w}_2 + (\lambda^2 - \gamma^2)\tilde{w}_3 = 0$$

The determinant of this system of equations for $\tilde{w}_2$ and $\tilde{w}_3$ must vanish

$$\lambda^2 - 2\lambda^2(\beta^2 + \gamma^2) = 0$$

Then

$$\lambda^2 = 0$$

and

$$\lambda^2 = \beta^2 + \gamma^2$$

The solutions for $\lambda^2 = \beta^2 + \gamma^2$ die out exponentially in the x-direction. They are harmless. Of greater interest are the solutions for $\lambda^2 = 0$. The occurrence of a double root is an

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indication that particular solutions of the form Eq. (277) do not exhaust all possibilities. Further information can be obtained by studying particular solutions, which are singular at some point, for instance $x = 0$, $y = 0$, $z = 0$, or $x = L$, $y = 0$, $z = 0$. For this purpose we introduce cylindrical coordinates x, r, θ

$$r^2 = y^2 + z^2$$

$$\theta = \text{arcty}(y/x)$$

and velocity component $w^{(r)}$ and $w^{(\theta)}$ in the directions of increasing r and θ respectively

$$w_2 = w^{(r)} \cos \theta - w^{(\theta)} \sin \theta$$

$$w_3 = w^{(r)} \sin \theta + w^{(\theta)} \cos \theta$$

It is practical to write the first of Eqs. (276) in the form

$$\partial^2 w_2/\partial x^2 + \frac{\partial}{\partial y} ((\partial w_2/\partial y) + (\partial w_3/\partial z)) = 0 \qquad (278)$$

The expression $\partial w_2/\partial y + \partial w_3/\partial z$ is the divergence in the y,z plane. Therefore,

$$\partial w_2/\partial y + \partial w_3/\partial z = ((\partial w^{(r)}/\partial r) + (w^{(r)} r^{-1})) + (\partial w^{(\theta)}/\partial \theta) r^{-1}$$

Then

$$\frac{\partial}{\partial y} [(\partial w_2/\partial y) + (\partial w_3/\partial z)]$$

$$= \{(\partial^2 w^{(r)}/\partial r^2 + \frac{\partial}{\partial r}[w^{(r)} r^{-1} + (\partial w^{(\theta)}/\partial \theta) r^{-1}]\} \cos \theta$$

$$- (\partial^2 w^{(\theta)}/\partial \theta^2) r^{-2} \sin \theta$$

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One thus obtains from Eq. (278)

$$(\partial^2 w^{(r)}/\partial x^2)\cos\theta - (\partial^2 w^{(\theta)}/\partial x^2)\sin\theta$$
$$+ \frac{\partial}{\partial r}[(\partial w^{(r)}/\partial r) + w^{(r)}r^{-1} + (\partial w^{(\theta)}/\partial\theta)r^{-1}]\cos\theta$$
$$- \frac{\partial}{\partial\theta}[(\partial w^{(r)}/\partial r) + w^{(r)}r^{-1} + (\partial w^{(\theta)}/\partial\theta)r^{-1}]r^{-1}\sin\theta = 0$$

Rather than transforming the second of Eqs. (276), we observe that the last equation must be correct for $\theta = 0$ and $\theta = \frac{\pi}{2}$. One then obtains the system

$$\partial^2 w^{(r)}/\partial x^2 + \frac{\partial}{\partial r}[(\partial w^{(r)}/\partial r) + w^{(r)}r^{-1} + (\partial w^{(\theta)}/\partial\theta)r^{-1}] = 0$$

and

$$(\partial^2 w^{(\theta)}/\partial x^2) + r^{-1}\frac{\partial}{\partial\theta}[\partial w^{(r)}/\partial r + w^{(r)}r^{-1}) + (\partial w^{(\theta)}/\partial\theta)r^{-1}] = 0$$

Restricting oneself to axisymmetric solutions ($w^{(r)}$ and $w^{(\theta)}$ independent of θ) one obtains

$$\partial^2 w^{(r)}/\partial x^2 + \partial^2 w^{(r)}/\partial r^2 + r^{-1}(\partial w^{(r)}/\partial r) - r^{-2}w^{(r)} = 0 \qquad (279)$$

and

$$\partial^2 w^{(\theta)}/\partial x^2 = 0 \qquad\qquad (280)$$

Eq. (279) is solved by a product

$$w^{(r)} = \exp(\pm\lambda x)Z_1(\lambda r)$$

where Z_1 denotes some linear combination of Bessel functions J_1 and N_1. It is important that these solutions die out in the x-direction. Of particular interest are solutions of Eq. (280)

$$w^{(\theta)} = w_1^{(\theta)} + w_2^{(\theta)} \cdot x$$

where $\tilde{w}_1^{(\theta)}$ and $\tilde{w}_2^{(\theta)}$ are constants. They pertain to the above solutions for $\lambda = 0$. The first part represents the vortices which in the linearized flow extend in constant strength along lines parallel to the x-axis, the second part represent the modification caused by a mismatch. By setting $\tilde{w}(r) = 0$, one can construct a solution which does not change the vortex distribution in the plane $x = 0$, while at $x = L$ it possesses a given axisymmetric vorticity. By combining expression with different origins for the cylindrical coordinates one can also construct more complicated examples which show the effect of misaligned vortices.

SECTION XI
DENSITY MODIFICATION

In flow fields containing supersonic regions further
measures are needed. They serve to control instabilities which
may arise in the supersonic region and to suppress expansion
shocks. Conceptually they are of a different nature, some
methods of residual minimization are stable in the supersonic
region, but all will admit expansion shocks. Regarding stability
we make the following observations. By their nature the Euler
equations obtained by residual minimization do not differentiate
between upstream and downstream, therefore, instabilities may
well arise. The methods are, however, of a global nature and
contain some implicitness, which may stabilize the procedure in
any case. The directional preference of supersonic flow is
introduced by the boundary conditions. In Sections XIII and XIV
we shall discuss specific examples.

The "damping" is frequently introduced by means of a
density modification. Here the following question arises. In
order for a potential flow to exist, it is necessary that the
density be a function of only the pressure. Density modification
seems to contradict this requirement. However, the existence of
a potential flow is compatible with the presence of sources
within the flow field (see Appendix B). Expressing the velocity
as gradient of a potential guarantees that the Euler equations
are satisfied, not that the flow field be source free. Density
modification is compatible with the concept of a potential flow,
because it can be interpreted as a judicious introduction of
sources. If density modification is confined to a restricted
region of the flow field, then the total strength of the sources
is zero, overall conservation of mass is maintained.

Density modification, if one expresses it in a non-
discretized form, raises the order of the differential equation.
We study its effect in some detail. Let ρ be the density in the
form as it is computed from the Bernoulli equation via the
pressure, and $\bar{\rho}$ the modified density. One sets

$$\tilde{\rho} = \rho + \epsilon(x,y,z) \text{ grad } \vec{w} \cdot (\vec{w}/|w|)$$

and replaces the equation of conservation of mass by

$$\text{div}(\tilde{\rho}\vec{w}) = 0$$

Then

$$\text{div}(\tilde{\rho}\vec{w}) = \text{div}(\rho\vec{w}) + \text{div}(\tilde{\rho}-\rho)\vec{w}$$

We study the effect for a linearized flow

$$\vec{w} = e_x U + \vec{W}$$

We had

$$\rho\vec{w} = \vec{m}(\vec{w})$$

$$\text{div}(\rho\vec{w}) = (1-M^2)W_{1x} + W_{1y} + W_{1z}$$

One then obtains

$$(1-M^2)W_{1x} + W_{1y} + W_{1z} + U \frac{\partial}{\partial x}\left(\epsilon \frac{\partial W_1}{\partial x}\right) = 0$$

The last term introduces the density modification with a coefficient ϵ which is presumably small. In this term second derivatives are encountered. We introduce a perturbation potential

$$W = \text{grad } \phi$$

and set immediately

$$\phi(x,y,z) = \exp(\lambda x)\exp(i(\beta y + \gamma z))$$

One then obtains

$$\lambda^2(1-M^2) - (\beta^2 + \gamma^2) + \epsilon M \lambda^3 = 0 \qquad (281)$$

Here U, made dimensionless with the velocity of sound of the basic flow, has been replaced by the Mach number M. For ϵ small one obtains in a first approximation the familiar linearized solution for subsonic flow

$$\lambda_o = \pm[(\beta^2 + \gamma^2)/(1-M^2)]^{1/2}$$

and for supersonic flow

$$\lambda_o = \pm[i(\beta^2 + \gamma^2)/(M^2-1)]^{1/2}$$

The effect of the density modification (introduced by ϵ) appears in the next approximation. Setting

$$\lambda = \lambda_o + \lambda_1$$

and considering λ_1 as $O(\epsilon)$, one obtains from Eq. (281)

$$2\lambda_o\lambda_1(1-M^2) + \epsilon M \lambda_o^3 = 0$$

$$\lambda_1 = -\epsilon \frac{M\lambda_o^2}{1-M^2} = -\epsilon(\beta^2 + \gamma^2)M(1-M^2)^{-2}$$

This gives

$$\phi = \exp(i(\beta y + \gamma z))\exp(\pm(\beta^2 + \gamma^2)^{1/2}(1-M^2)^{-1/2})$$

$$\times \exp(-\epsilon(\beta^2 + \gamma^2)^2(1-M^2)^{-2}Mx)$$

For both subsonic and supersonic Mach number the presence of ϵ introduces a factor, which causes the solution to decrease exponentially in the flow direction (here the x-direction).

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There exists, however, a third solution of Eq. (281). It
is found if we write it in the form

$$\epsilon M \lambda_o + (1-M^2) - (\beta^2 + \gamma^2)/\lambda_o^2 = 0$$

The first approximation is obtained from the first and second
term

$$\lambda_o = -\epsilon^{-1}(1-M^2)M^{-1} \tag{282}$$

For small ϵ the third term is indeed negligible. Writing again

$$\lambda = \lambda_o + \lambda'$$

one finds

$$\lambda' = \epsilon(\beta^2 + \gamma^2)M^2(1-M^2)^{-2}$$

the correction expressed by λ' is very small. The main part λ_o
is independent of β and γ. It is negative for subsonic and
positive for supersonic flows, it gives solutions which decrease
or increase exponentially very quickly if ϵ is small. It is
controlled by a boundary condition at the beginning of a subsonic
region, or at the end of a supersonic region. In some form these
boundary conditions will appear also in a discretized version.

Regarding the exclusion of shocks, we follow an idea of
Glowinski-Pironneau (Reference 5). They are quoted in Section
2.2.4 and 5.2 in Reference 2.

One has from conservation of mass

$$\text{div}(\rho \vec{w}) = 0$$

This holds even in shocks, if one interprets this equation in the
sense of generalized solutions. Hence from general vector
relation

$$\rho \text{ div } \vec{w} + \text{grad } \rho \cdot \vec{w} = 0$$

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or

$$\text{div } \vec{w} = -\text{grad } (\log \rho) \cdot \vec{w}$$

In an accelerating flow the density decreases in the flow direction, even in an expansion shock. Therefore,

$$\text{div } \vec{w} > 0$$

To suppress flows with two strong an expansion one will postulate

$$(\text{div } \vec{w} - c) < 0$$

where c is some nonpositive constant. In the minimization process this can be included by adding to the expressions J a penalty function

$$\int_{\Omega} c_1(x,y,z)[(\text{div } \vec{w} - c)^+]^2 d\tau$$

where

$$(\text{div } \vec{w} - c)^+ = \text{supr}[0, \text{div } \vec{w} - c]$$

and $c_1(x,y,z)$ is positive or zero. It seems possible to choose $c_1 \equiv 0$ except in the region of an incipient rarefraction shock. For simplicity one may choose the threshold constant c zero. Whether this, or other forms of biasing against expansions shocks, is preferable requires numerical experimentation. (Some experiences are reported in Reference 2, and it is likely that these authors have meanwhile added further studies.)

The effect of density modification can be studied in the one-dimensional problem. We consider the flow through a Laval nozzle whose area is given by

$$A(x) = 1 + x^2$$

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The throat lies then at x = 0. The velocity vector has only an x
component denoted by u(x), made dimensionless with the sonic
velocity. For a gas with ratio of the specific heat γ = 1.4 one
has

$$\rho = (1.2 - 0.2u^2)^{2.5}$$

and

$$m(u) = u(1.2 - 0.2u^2)^{2.5}$$

With density modification one has

$$\tilde{\rho} = \rho + \epsilon \frac{\partial u}{\partial x}$$

Actually, we have set

$$\tilde{m}(u) = m(u) + \tilde{\epsilon} \frac{\partial u}{\partial x}$$

One has

$$\tilde{\epsilon} = u\epsilon$$

In the example we have assumed $\tilde{\epsilon}$ = const = .1. This means that
we consider ϵ as changing with u. This does not affect the
character of the curves. Actually the value of $\tilde{\epsilon}$ chosen here is
too large for practical purposes. In reality ϵ will be taken
proportional to the mesh width in the x-direction. Here we have
found the results by integrating the resulting differential
equation. This equation is stiff (because ϵ is small). The
velocity distribution is determined by the equation of
conservation of mass. One has immediately

$$A\tilde{m} = const = A_o m_o$$

where the constant represents the total mass flow. Then

$$\frac{du}{dx} = \epsilon^{-1}\left(\frac{A_o}{A(x)} m_o - m(u)\right) \qquad (283)$$

Without density modification one has

$$m(u)A(x) = A_o m_o$$

With the simple form for A chosen above one obtains

$$x = \left(\frac{A_o m_o}{m(u)} - 1\right)^{1/2} \qquad (284)$$

The result shown in Fig. 10 is of course familiar. It shows u versus x for constant values of $Am = (A_o m_o)$. The curves $Am = 1$ divide the area into four quadrants. The one going from the lower left to the upper right gives the transition from subsonic to supersonic at the throat ($x = 0$). Only for $Am \leq 1$ does one obtain a flow through the muzzle, for $Am < 1$ it is either entirely subsonic or entirely supersonic. The curves for $Am > 1$ have no physical meaning.

One can have jumps at constant x in the right half of the figure from the supersonic curve $Am = 1$ to the subsonic curve $Am = 1$. These are shocks if one neglects the effect of the entropy change. A comparison of some such curves for subsonic flows with the curves resulting from Eq. (284) is shown in Fig. 11. The curves start at the same value of u for the entrance cross section $x = -1$. At this point the right-hand side of Eq. (283) vanishes and $du/dx = 0$. The curves have about the same character. The deviations could be made smaller if one would use a smaller value of ϵ. In any case, with boundary conditions given at the left, modification of the density gives a stable procedure in the subsonic region. This is in accordance with Eq. (282).

A more extended survey is shown in Fig. 12. As soon as the curves reach the supersonic region ($u > 1$), they deviate very

quickly from the ideal curves. There are curves for which u remains supersonic and increases until one reaches the maximum possible velocity. This part is controlled by ϵ and not by the expression for $\epsilon = 0$. These are, of course, without physical interest.

There is a second set of curves which leads with increasing x to smaller velocities. Once one attains subsonic velocities the tendency of the curves to come close to an ideal curve (shown by Eq. (282)) makes itself felt, and the curves approach a smooth subsonic flow. In this manner, one obtains the picture of a smoothed out shock. Different shock locations are obtained by varying the conditions at $x = -1$. But the changes are so small that the curves for different shock locations seem to arise and also terminate at one curve. One recognizes how the concept of density modification leads to the representation of shocks (in smeared out form), expansion shocks cannot arise, because, in the subsonic region the term for density modification does not allow a curve to "break away" from the "smooth" solution.

SECTION XII
PROPERTIES OF DIFFERENT METHODS, AN OVERVIEW

In this section the salient properties of the methods discussed above are summarized.

a.   Sensitivity of J against short wave errors.   The wave length of perturbations is proportional to λ^{-1}.   The perturbations are assumed to be of equal magnitude in the velocities.

$\vec{w}$ method λ^{-4}

$\vec{A},\phi$ method λ^{-2}

ϕ,ξ method λ^{-2}

All three methods give (in the limit toward a small mesh) legitimate minimization problems if one uses a Bristeau-like distance definition in the function space.

b.   Shock Capturing.

$\vec{w}$ method   not feasible, (except perhaps in a coarse grid)

$\vec{A},\phi$ method   feasible

ϕ,ξ method   feasible

c.   Shock Identification.

$\vec{w}$ method   feasible (and necessary after preliminary coarse grid steps)

$\vec{A},\phi$ method   feasible

ϕ,ξ method   feasible, (but expensive because of the necessity of repeated evaluation of ξ)

d.   Accuracy for elements of the same size and shape functions of the same character.

$\vec{w}$ method   gives higher accuracy than the $\vec{A},\phi$ and ϕ,ξ methods because of the differentiations requires for the determination of the velocities and the pressures.

e. Determination of J. (needed in the search procedure)

$\vec{w}$ and $\vec{A}, \phi$ methods   the determination of J does not require the solution of a partial differential equation.

ϕ, ξ method   for the determination of J the solution of a partial differential equation is necessary.

f. Multigrid method in conjuction with a simple-minded distance definition.

$\vec{w}$ and $\vec{A}, \phi$ methods   feasible

ϕ, ξ method   not feasible

g. Multigrid Methods with Bristeau-like metric.

Feasible in all cases.

h. Wake Capturing.

Feasible only in the $\vec{w}$ method.

i. Evaluation of the inhomogeneous term occuring in the determination of the Gradient function.

By evaluation of integrals: feasible in all methods.

By numerical differentiation: not feasible in the ϕ, ξ method.

j. Number of Poisson solutions required in the gradient determination.

Two-dimensional problems:

$\vec{w}$ method   2

ϕ, ψ method   2

ϕ, ξ method   1

Three-dimensional problems:

$\vec{w}$ method   3

$\vec{A}, \phi$ method   4

ϕ, ξ method   1

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Judging according to the last item, the ϕ, ξ method might appear preferable to the other possibilities. However, each of the methods requires for the search procedure repeated evaluations of J, and for the ϕ, ξ method this entails the repeated evaluation of ξ by means of a Poisson solver. One might, therefore, consider the $\vec{w}$ method as preferable, although it is not well suited for shock capturing. It is an advantage of the $\vec{w}$ method that one can use larger elements if one wants to attain the same accuracy as with other methods. This may compensate for the additional labor of identifying the shock position.

Of course, numerical experimentation may uncover points of view which invalidate these conclusions.

SECTION XIII
LINEARIZED SUPERSONIC FLOW TREATED IN
DISCRETIZED FORM BY THE BRISTEAU METHOD

In the next two Sections the discretized treatment of a supersonic flow linearized for the vicinity of a parallel flow will be investigated. This investigation was initiated in order to see how the method works in an example which does not seem to be particularly well suited for residual minimization. Beyond this, one may draw some inferences, although not quite directly, for the behavior of solutions in a supersonic region embedded in a subsonic flow and also for time-dependent problems. The procedure has some similarity with the derivation of von Neumann stability conditions, except that here the results arise from a global formulation. The examples will show, that artificial damping is not always necessary.

One may think of two realizations of a discretized method of residual minimization. In the first one one would start with the discretized flow equations and apply to them the idea of residual minimization. Here one solves the flow equations in the discretized form, and the stability is of course determined by these equations; the method by which they are solved is immaterial. Alternatively one may first express the functional J in a discretized form, for instance by means of a finite element representation of the dependent variables and afterwards minimize J by choosing the parameters which determine it. There should not be significant differences between the two approaches especially for long waves, but in a supersonic flow small modifications may still cause a transition from a procedure which is neutrally stable to one which is unstable. Even the second formulation will, however, be compatible with expansion shocks; some measure which discriminates against expansion shocks will always be necessary.

One needs particular solutions suited to solve well-posed boundary value problems. In a supersonic flow one must differentiate between upstream and downstream; one has in

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particular an entrance cross section, and, because of the global nature of the residual minimization procedure an exit cross section. These are boundaries of a space-like character. Usually one has also conditions along side walls of a time-like character, but for the present work they will be replaced by periodicity conditions.

We shall consider the linearized problem, that is, small perturbations of a supersonic parallel flow with an angle θ of the velocity vector with the x-axis. The problem is treated in a finite element approach with rectangular elements oriented in the x and y directions. The element dimensions are Δx and Δy. A simple situation arises if $\theta = 0$; the flow is aligned with the grid system. A well-posed problem is then obtained if the entrance and the exit cross section are given by lines x = const. This, however, is too special to be instructive. We shall allow for the following situation. Beside the x,y system we introduce a Cartesian system ξ, η in which the ξ-axis is aligned with the velocity vector. Moreover, the entrance and exit cross section (assumed to be straight lines parallel to each other) need not be perpendicular to the x- or to the ξ-axis. The normal to these cross sections forms an angle ν with the x-axis. We introduce a third Cartesian system ξ_1, η_1 where the ξ_1-axis has the direction normal to the entrance and exit cross section. Since these cross sections must be space-like one has

$$|(\nu - \theta)| < (\pi/2) - \alpha$$

where α is the Mach angle ($\sin \alpha = M^{-1}$). One has the following relations

$$\xi = x \cos \theta + y \sin \theta$$

$$\tag{285}$$

$$\eta = -x \sin \theta + y \cos \theta$$

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$$\xi_1 = \xi \cos(\nu-\theta) + \eta \sin(\nu-\theta)$$

$$\eta_1 = -\xi \sin(\nu-\theta) + \eta \cos(\nu-\theta) \tag{286a}$$

$$\xi = \xi_1 \cos(\nu-\theta) - \eta_1 \sin(\nu-\theta)$$

$$\eta = \xi_1 \sin(\nu-\theta) + \eta_1 \cos(\nu-\theta) \tag{286b}$$

$$\xi_1 = x \cos \nu + \eta \sin \nu$$

$$\eta_1 = -x \sin \nu + \eta \cos \nu \tag{287}$$

Exact particular solutions are most readily expressed in the ξ, η system, viz

$$\phi = \bar{\phi} \exp(i\bar{\omega}(\cos \alpha)^{-1}(\pm\xi \sin \alpha + \eta \cos \alpha)$$

Hence, with Eqs. (285) and (286a)

$$\phi = \bar{\phi} \exp(i\bar{\omega}(\cos \alpha)^{-1}(-x \sin(\theta\mp\alpha) + y \cos(\theta\mp\alpha)) \tag{288}$$

and

$$\phi = \bar{\phi} \exp(i\bar{\omega}(\cos \alpha)^{-1}(-\xi_1 \sin(\theta-\nu\mp\alpha) + \eta_1(\cos(\theta-\nu\mp\alpha)) \tag{289}$$

The signs in Eqs. (288) and (289) must be used consistently. The upper and lower signs give, respectively, right- and left-going single waves. We shall try to find particular solutions corresponding to such single waves.

The exact solutions are actually sinusoidal in any direction. To be able to satisfy arbitrary initial conditions along the entrance cross section given by $\xi_1 = $ const, the particular solutions must be sinusoidal in the η_1 direction, also in the discretized treatment. One cannot guarantee that after the discretization the solutions will be sinusoidal also in the ξ_1 direction, but they will be of course expressible in terms of

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exponential functions. We try to find solutions of the form (Eq. (289))

$$\phi = \tilde{\phi} \exp(i(\omega_1 \xi_1 + \omega_2 \eta_1)) \qquad (290)$$

where ω_1 is real. The computation is carried out in the x,y system. With Eq. (287), the last expression transforms into

$$\phi = \tilde{\phi} \exp(i\omega_3 x + \omega_4 y) \qquad (291)$$

with

$$\omega_3 = \omega_1 \cos \nu - \omega_2 \sin \nu$$
$$\omega_4 = \omega_1 \sin \nu + \omega_2 \cos \nu \qquad (292)$$

The computation will give a relation between ω_3 and ω_4 (ω_3 and ω_4 may be complex, in principle at least). For solutions of interest, one has the requirement that

$$\omega_2 = -\omega_3 \sin \nu + \omega_4 \cos \nu \qquad (293)$$

is real.

The Bristeau method uses simultaneous two variables, namely ϕ and ξ connected by the equation

$$\Delta \xi = \text{div } \vec{m}(\text{grad } \phi) \qquad (294)$$

This equation is treated by postulating

$$(1/2)\int_\Omega (\text{grad } \xi - \vec{m}(\text{grad } \phi))^2 d\tau = \text{Min}$$

Here the function that is to be varied is ξ. This leads to

$$\int_\Omega (\text{grad } \xi - \vec{m}(\text{grad } \phi)) \cdot \delta(\text{grad } \xi) d\tau = 0 \qquad (295)$$

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The functional J is expressed in terms of ζ (see Section V). The postulate that under variations of ϕ the functional J is minimized leads to the condition

$$\int_\Omega \text{grad } \zeta \cdot (M\delta \text{ grad } \phi)d\tau = 0 \qquad (296)$$

The matrix M is defined in Eq. (33). In the coming discussions we shall write N instead of M and reserve M for the Mach number. We consider the linearized form of N under the assumption that the flow direction forms an angle with the x-axis. Then one has for a two-dimensional flow

$$N = \begin{pmatrix} (1-M^2 \cos^2 \theta) & -M^2 \sin \theta \cos \theta \\ -M^2 \sin \theta \cos \theta & (1-M^2 \sin^2 \theta) \end{pmatrix} \qquad (297)$$

Eq. (296) is subject to boundary conditions which ensure that $\zeta \equiv 0$. This important simplification occurs in the discretized as well as in the nondiscretized formulation. In other approaches a corresponding simplification will not be encountered. Eq. (295) therefore simplifies to

$$\int_\Omega \vec{m}(\text{grad } \phi) \cdot \delta \text{ grad } \zeta \, d\tau = 0 \qquad (298)$$

For the linearized case considered here one then obtains

$$\int_\Omega (N \text{ grad } \phi) \cdot \delta \text{ grad } \zeta \, d\tau = 0$$

It is natural to represent ϕ and ζ by the same shape functions. Writing the matrix N in detail and replacing δ grad ϕ by the shape function χ_{mn} (defined later) one obtains from the last equation

$$a_1 \int_\Omega \phi_x \chi_{mn,x} \, dxdy + a_2 \int_\Omega \phi_y \chi_{mn,x} \, dxdy + a_2 \int_\Omega \phi_x \chi_{mn,y} \, dxdy$$

$$\tag{299}$$

$$+ \, a_3 \int_\Omega \phi_y \chi_{mn,y} \, dxdy = 0$$

with

$$a_1 = (1 - M^2 \cos^2 \theta)$$

$$a_2 = -M^2 \sin \theta \cos \theta \tag{300}$$

$$a_3 = (1 - M^2 \sin^2 \theta)$$

We assume that the element boundaries form a rectangular grid oriented in the x and y directions. The grid points are numbered by two subscripts (usually m and n), the first one for the x and the second one for the y direction. The grid spacings are denoted by Δx and Δy. The elements are given the numbers of the grid point at the lower left. In the present examples we use bilinear shape functions. The shape parameters are the values of ϕ at the grid points. The value of ϕ at a grid point (m,n) is denoted by ϕ_{mn}. The shape function χ_{mn} is bilinear in the elements surrounding the point (m,n) otherwise is zero. Therefore, one has

$$\chi_{mn}(x,y) = \Delta x^{-1} \Delta y^{-1} (x - x_{m-1})(y - y_{n-1}) \quad \text{in the element } (m-1, n-1);$$

$$\chi_{mn}(x,y) = -\Delta x^{-1} \Delta y^{-1} (x - x_{m+1})(y - y_{n-1}) \quad \text{in the element } (m, n-1);$$

$$\tag{301}$$

$$\chi_{mn}(x,y) = \Delta x^{-1} \Delta y^{-1} (x - x_{m+1})(y - y_{m+1}) \quad \text{in the element } (m,n)$$

$$\chi_{mn}(x,y) = -\Delta x^{-1} \Delta y^{-1} (x - x_{m-1})(y - y_{n+1}) \quad \text{in the element } (m-1, n)$$

$$\chi_{mn}(x,y) = 0 \text{ everywhere else.}$$

Then one has as representation of $\phi(x,y)$ in the element $(m-1, n-1)$

$$\phi(x,y) = \phi_{m-1,n-1}\,\chi_{m-1,n-1} + \phi_{m-1,n}\,\chi_{m-1,n}$$

$$+ \phi_{m,n-1}\,\chi_{m,n-1} + \phi_{m,n}\,\chi_{m,n}$$

As mentioned above, the ϕ_{jk}'s are the shape parameters. The dependence upon x and y enters through the shape function χ_{jk}. Corresponding expressions arise for the other elements surrounding the point (m,n). One finds for the integrals occurring in Eq. (299)

$$\int_{\Omega} \phi_x \chi_{mn,x}\, dxdy =$$

$$(\Delta y/\Delta x)[-(1/6)\phi_{m-1,n-1} + (1/3)\phi_{m,n-1} - (1/6)\phi_{m+1,n-1}$$
$$-(2/3)\phi_{m-1,n} + (4/3)\phi_{m,n} - (2/3)\phi_{m+1,n} \qquad (302)$$
$$-(1/6)\phi_{m-1,n+1} + (1/3)\phi_{m,n+1} - (1/6)\phi_{m+1,n+1}]$$

$$\int_{\Omega} \phi_x \chi_{mn,y}\, dxdy = \int_{\Omega} \phi_y \chi_{mn,x}\, d\tau = \qquad (303)$$

$$[-(1/4)\phi_{m-1,n-1} + (1/4)\phi_{m+1,n-1} + (1/4)\phi_{m-1,n+1} - (1/4)\phi_{m+1,n+1}]$$

$$\int_{\Omega} \phi_y \chi_{mn,y}\, dxdy =$$

$$(\Delta x/\Delta y)[-(1/6)\phi_{m-1,n-1} - (2/3)\phi_{m,n-1} - (1/6)\phi_{m+1,n-1}$$
$$+(1/3)\phi_{m-1,n} + (4/3)\phi_{m,n} + (1/3)\phi_{m+1,n} \qquad (304)$$
$$-(1/6)\phi_{m-1,n+1} - (2/3)\phi_{m,n+1} - (1/6)\phi_{m+1,n+1}]$$

By substituting these expressions into Eq. (299) one obtains relations between the shape parameters ϕ_{jk}, but this substitution is postponed.

We set in accordance with Eq. (291)

$$\phi_{mn} = \tilde{\phi}\, \exp(i(\omega_3 m\Delta x + \omega_4 n\Delta y) \qquad (305)$$

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where $\tilde{\phi}$ is a constant. One obtains by substitution into
Eqs. (302), (303), and (304)

$$\int_\Omega \phi_x X_{mn,x} dxdy = \tilde{\phi} \exp(i(m\omega_3 \Delta x + \omega_4 \Delta y)) \qquad (306)$$

$$(\Delta y/\Delta x)2(1-\cos(\omega_3 \Delta x))((2/3) + (1/3)\cos(\omega_4 \Delta y))$$

$$\int_\Omega \phi_x X_{mn,y} dxdy = \int_\Omega \phi_y X_{mn,x} dxdy = \qquad (307)$$

$$\tilde{\phi} \exp(i(m\omega_3 \Delta x + \omega_4 \Delta y))\sin(\omega_3 \Delta x)\sin(\omega_4 \Delta y)$$

$$\int_\Omega \phi_y X_{mn,y} dxdy = \tilde{\phi} \exp(i(m\omega_3 \Delta x + \omega_4 \Delta y) \qquad (308)$$

$$(\Delta x/\Delta y)2((2/3) +(1/3)\cos(\omega_3 \Delta x))(1-\cos(\omega_4 \Delta y))$$

At the moment one does not know whether ω_3 and ω_4 will be
real. One obtains by substitution into Eq. (299)

$$a_1(\Delta y/\Delta x)(1-\cos(\omega_3 \Delta x))((4/3) + (2/3)\cos(\omega_4 \Delta y)$$

$$+ 2a_2 \sin(\omega_3 \Delta x)\sin(\omega_4 \Delta y) \qquad (309)$$

$$+ a_3(\Delta x/\Delta y)((4/3) + (2/3)\cos(\omega_3 \Delta x)(1-\cos(\omega_4 \Delta y) = 0$$

For given M^2, θ, Δx, Δy, and real ω_4, one can solve this equation
in the following manner. Let

$$\exp(i\omega_3 \Delta x) = r \qquad (310)$$

Then one obtains

$$a_1(\Delta y/\Delta x)((2/3) + (1/3)\cos(\omega_4 \Delta y)((-r^{-1} + 2 \qquad - r)$$

$$+ a_2 \qquad\qquad \sin(\omega_4 \Delta y) \qquad (-ir^{-1} \qquad\qquad + ir)$$

$$+ a_3(\Delta x/\Delta y)(1-\cos(\omega_4 \Delta y)) \qquad\qquad ((1/3)r^{-1} + (4/3) + (1/3)r)$$

This is a quadric equation for r of the form

$$(a + ib)r^2 + 2cr + (a-ib) = 0$$

Then

$$r = (a + ib)^{-1}(-c + (c^2 - (a^2 + b^2))^{1/2}$$

For

$$a^2 + b^2 - c^2 > 0 \qquad\qquad (311)$$

one has

$$r = (a + ib)^{-1}(-c \pm i(a^2 + b^2 - c^2)^{1/2})$$

Therefore

$$|r| = 1$$

It follows from Eq. (310) that

$$\omega_3 \text{ real} \qquad \text{for } a^2 + b^2 - c^2 > 0.$$

Let

$$\bar{a}_1 = a_1((2/3) + (1/3)\cos(\omega_4 \Delta y))$$

$$\bar{a}_2 = a_2 \sin(\omega_4 \Delta y) \qquad\qquad (312)$$

$$\bar{a}_3 = a_3(1/3)(1 - \cos(\omega_4 \Delta y))$$

Then one has

$$a = -(\Delta y/\Delta x)\bar{a}_1 + (\Delta x/\Delta y)\bar{a}_3$$

$$b = + \bar{a}_2 \qquad\qquad (313)$$

$$c = (\Delta y/\Delta x)\bar{a}_1 + (\Delta x/\Delta y)2\bar{a}_3$$

Then from the inequality (Eq. (311))

$$(-(\Delta y/\Delta x)\bar{a}_1 + (\Delta x/\Delta y)\bar{a}_3)^2$$

$$+ \bar{a}_2{}^2 - ((\Delta y/\Delta x)\bar{a}_1 + (\Delta x/\Delta v)2\bar{a}_3)^2 > 0$$

Hence

$$(\Delta x/\Delta y)^2 < (\bar{a}_2{}^2 - 6\bar{a}_1\bar{a}_3)/(3\bar{a}_3{}^2) \qquad\qquad (314)$$

Thus, Eq. (314) gives a limit for the ratio $(\Delta x/\Delta y)$ for which ω_3 is real. If this condition is not satisfied, then there is always one solution for which $|r| > 1$, because the product of the two roots r is given by $(a - ib)/(a + ib)$ and

$$|(a - ib)/(a + ib)| = 1.$$

This discussion can be carried further for $\theta = 0$. Then one obtains

$$(1-M^2)(\Delta y/\Delta x)((2/3) + (1/3)\cos(\omega_4\Delta y))(-r^{-1} + 2 - r)$$

$$+ (\Delta x/\Delta y)(1 - \cos(\omega_4\Delta y))((1/3)r^{-1} + (4/3) + (1/3)r)$$

Then r depends only upon

$$\left(\frac{\Delta y}{\Delta x}\right)(M^2-1)^{1/2} = Cr^{-1} \qquad\qquad (315)$$

where Cr denotes the Courant number of the mesh. The limit for the mesh ratio is given by

$$(\Delta x/\Delta y)^2 \leq (M^2-1)(4 + 2\cos(\omega_4 \Delta y)/(1 - \cos(\omega_4 \Delta y))$$

With the definition Eq. (315) one can write

$$Cr^2 < (4 + 2\cos(\omega_4 \Delta y))/(1 - \cos(\omega_4 \Delta y)) \qquad (316)$$

This limit for the Courant number ranges from ∞ for $(\omega \Delta y) = 0$, over 2 for $(\omega_4 \Delta y) = (\pi/2)$ to 1 for $\omega_4 \Delta y = \pi$. One has a stable scheme if the mesh ratio is sufficiently small, so that no special measures as upwinding or density modifications are necessary.

For $\theta = 0$ and $Cr = 1$, $a_1 = -(M^2-1)$, $a_2 = 0$, $a_3 = 1$ and one obtains from Eq. (309)

$$-(1 - \cos(\omega_3 \Delta x)((4/3) + (2/3)\cos(\omega_4 \Delta y))$$

$$+(4/3) + (2/3)\cos(\omega_3 \Delta x)(1 - \cos(\omega_4 \Delta y)) = 0$$

This is obviously solved by

$$\omega_3 \Delta x = \pm \omega_4 \Delta y$$

Hence (since $Cr = 1$),

$$\omega_3 = \pm \omega_4 (M^2-1)^{-1/2}$$

The expression (Eq. (305) then assumes the form

$$\phi_{mn} = \tilde{\phi}(\exp(i\omega_4(\pm(M^2-1))^{-1/2} m\Delta x + n\Delta y)$$

At the mesh points one therefore obtains the exact solution.

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SECTION XIV

LINEARIZED TWO-DIMENSIONAL SUPERSONIC FLOWS TREATED BY THE ϕ, ψ METHOD

The nondiscretized problem is discussed first to provide a guide for the results of the discretized procedure. The expression to be minimized in the two-dimensional case is given by

$$J = (1/2) \int_{\Omega} (\vec{m}(\text{grad } \phi) - \vec{e}_x \psi_y + \vec{e}_y \psi_x)^2 dxdy$$

As in the preceding section, $\vec{m}(\text{grad } \phi)$ is replaced by

$$N \text{ grad } \phi = \begin{pmatrix} (1 - M^2 \cos^2\theta) & -M^2 \sin\theta \cos\theta \\ -M^2 \sin\theta \cos\theta & 1 - M^2 \sin^2\theta) \end{pmatrix} \begin{pmatrix} \phi_x \\ \phi_y \end{pmatrix}$$

Thus one has

$$J = (1/2) \int_{\Omega} \{[(1 - M^2 \cos^2\theta)\phi_x - (M^2 \sin\theta \cos\theta)\phi_y - \psi_y]^2$$

$$+ [-(M^2 \sin\theta \cos\theta)\phi_x + (1 - M^2 \sin^2\theta)\phi_y + \psi_x]^2\} dxdy$$

(317)

The terms in the brackets are the residuals. One obtains as Euler equations arising from the variation of ϕ

$$((1 - M^2 \cos^2\theta)\frac{\partial}{\partial x} - M^2 \sin\theta \cos\theta \frac{\partial}{\partial y})$$

$$[(1 - M^2 \cos^2\theta)\phi_x - M^2 \sin\theta \cos\theta \, \phi_y - \psi_y] \qquad (318)$$

$$+ (-M^2 \sin\theta \cos\theta \frac{\partial}{\partial x} + (1 - M^2 \sin^2\theta)\frac{\partial}{\partial y})$$

$$[-M^2 \sin\theta \cos\theta \, \phi_x + (1 - M^2 \sin^2\theta)\phi_y + \psi_x'] = 0$$

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and from the variation of ψ

$$\frac{\partial}{\partial y}[(1 - M^2 \cos^2\theta)\phi_x - (M^2 \sin\theta \cos\theta)\phi_y - \psi_y]$$

$$+ \frac{\partial}{\partial x}[-(M^2 \sin\theta \cos\theta)\phi_x + (1 - M^2 \sin^2\theta)\phi_y + \psi_x] = 0 \quad (319)$$

At the moment we assume that the entrance and exit cross sections are lines $x = 0$ and $x = L$ and that the solution is sinusoidal in the y direction. (In part of the further analysis this condition will not be imposed.) One then obtains as boundary terms

$$\{(1 - M^2 \cos^2\theta)[(1 - M^2 \cos^2\theta)\phi_x - (M^2 \sin\theta \cos\theta)\phi_y - \psi_y]$$

$$- M^2 \sin\theta \cos\theta [-(M^2 \sin\theta \cos\theta)\phi_x$$

$$+ (1 - M^2 \sin^2\theta)\phi_y + \psi_x]\}\delta\psi \Big|_{x=0}^{x=L} = 0$$

and

$$[-(M^2 \sin\theta \cos\theta)\phi_x + (1 - M^2 \sin^2\theta)\phi_y + \psi_x]\delta\psi \Big|_{x=0}^{x=L} = 0$$

For the entrance cross section, $x = 0$, ϕ and ψ are prescribed, therefore, $\delta\phi = \delta\psi = 0$. At the exit cross section, $x = L$, no conditions are prescribed in the original flow equations $\delta\phi$ and $\delta\psi$ are different from zero. Therefore, the residuals (given in the last equation by the terms in brackets) must vanish.

The Euler equations will certainly be satisfied, if the residuals vanish everywhere.

$$(1 - M^2 \cos^2\theta)\phi_x - (M^2 \sin\theta \cos\theta)\phi_y - \psi_y = 0$$

$$-(M^2 \sin\theta \cos\theta)\phi_x + (1 - M^2 \sin^2\theta)\phi_y + \psi_y = 0$$

Particular solutions are best obtained in the ξ, η system (introduced in Section XIII) for which the ξ-axis is aligned with

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the flow direction. The transformed equations are obtained simply by setting $\theta = 0$ in the last equations

$$(1 - M^2)\phi_\xi - \psi_\eta = 0$$

$$\phi_\eta + \psi_\xi = 0$$

A slight simplification arises if one sets

$$\xi = \bar{\xi}(M^2 - 1)^{1/2} = \bar{\xi}\cot\alpha; \quad \eta = \bar{\eta}$$

$$\phi(\xi, \eta) = \bar{\phi}(\bar{\xi}, \bar{\eta})$$

$$\psi(\xi, \eta) = (M^2 - 1)^{1/2}\bar{\psi}(\bar{\xi}, \bar{\eta})$$

This yields

$$\bar{\phi}_{\bar{\xi}} + \bar{\psi}_{\bar{\eta}} = 0$$

$$\bar{\phi}_{\bar{\eta}} + \bar{\psi}_{\bar{\xi}} = 0$$

with the obvious particular solutions

$$\bar{\phi} = \exp(i\omega(\mp\bar{\xi} + \bar{\eta}))$$

$$\bar{\psi} = \pm\exp(i\omega(\mp\bar{\xi} + \bar{\eta}))$$

These expressions can, of course, be transformed back into the x,y system. These are not the only particular solutions that are periodic in the $\bar{\eta}$ direction. In the ξ, η system the complete Euler equations are given by

$$(1 - M^2)\frac{\partial}{\partial\xi}[(1 - M^2)\phi_\xi - \psi_\eta] + \frac{\partial}{\partial\eta}[\phi_\eta + \psi_\xi] = 0$$

$$-\frac{\partial}{\partial\eta}[(1 - M^2)\phi_\xi - \psi_\eta] + \frac{\partial}{\partial\xi}[\phi_\eta + \psi_\xi] = 0$$

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Again the coordinates $\bar{\xi}$ and $\bar{\eta}$, and the dependent variables $\bar{\phi}$ and $\bar{\psi}$ are introduced. One obtains

$$(M^2 - 1)v_{1,\bar{\xi}} + v_{2,\bar{\eta}} = 0$$

$$(M^2 - 1)v_{1,\bar{\eta}} + v_{2,\bar{\xi}} = 0 \qquad (320)$$

with

$$v_1 = \bar{\phi}_{\bar{\xi}} + \bar{\psi}_{\bar{\eta}}$$

$$v_2 = \bar{\phi}_{\bar{\eta}} + \bar{\psi}_{\bar{\xi}} \qquad (321)$$

One may be surprised by the occurrence of the factor $(M^2 - 1)$ in these equations. It can be eliminated by assigning in the definition of the functional J different weights to the ξ and η direction. Such an adjustment is not made here, because it cannot be made in a more general setting. The functions v_1 and v_2 are the residuals.

With constants of integration c_1 and c_2 , one obtains

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ M^2 - 1 \end{pmatrix} \exp(i\omega(-\bar{\xi} + \bar{\eta})) + c_2 \begin{pmatrix} 1 \\ -(M^2 - 1) \end{pmatrix} \exp(i\omega(\bar{\xi} + \bar{\eta}))$$

This shows that there will be particular solutions of the Euler equations for which the residuals do not vanish. With Eqs. (321) one then obtains

$$\bar{\phi}_{\bar{\xi}} + \bar{\psi}_{\bar{\eta}} = c_1 \exp(i\omega(-\bar{\xi} + \bar{\eta})) + c_2 \exp(i\omega(\bar{\xi} + \bar{\eta}))$$

$$\bar{\phi}_{\bar{\eta}} + \bar{\psi}_{\bar{\xi}} = c_1(M^2 - 1)\exp(i\omega(-\bar{\xi} + \bar{\eta})) - c_2(M^2 - 1)\exp(i\omega(\bar{\xi} + \bar{\eta}))$$

Particular solutions which are periodic in the $\bar{\eta}$ direction are found by setting

$$\begin{pmatrix} \bar{\phi}(\bar{\xi},\bar{\eta}) \\ \bar{\psi}(\bar{\xi},\bar{\eta}) \end{pmatrix} = \begin{pmatrix} \tilde{\phi}(\xi) \\ \tilde{\psi}(\xi) \end{pmatrix} \exp(i\omega\bar{\eta})$$

One obtains a system of ordinary differential equations for $\tilde{\phi}(\xi)$ and $\tilde{\psi}(\xi)$. The general solution is given by

$$\begin{pmatrix} \bar{\phi}(\bar{\xi},\bar{\eta}) \\ \bar{\psi}(\bar{\xi},\bar{\eta}) \end{pmatrix} = \tag{322}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \{ c_1 (M^2/2)\bar{\xi} \exp(i\omega(-\bar{\xi}+\bar{\eta}) + c_2 i((M^2-2)/4\omega)\exp(i\omega(\bar{\xi}+\bar{\eta})$$

$$+ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \{ c_1 i((M^2-2)/4\omega)\exp(i\omega(-\bar{\xi}+\bar{\eta}) + c_2 (M^2/2)\bar{\xi} \exp(i\omega(\bar{\xi}+\bar{\eta}))$$

$$+ c_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp(i\omega(-\bar{\xi}+\bar{\eta})) + c_4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp(i\omega(\bar{\xi}+\bar{\eta}))$$

The boundary conditions at an exit cross section given by a line $\bar{\xi}$ = const are satisfied if $c_1 = c_2 = 0$; then only the solutions of the linearized flow equations (with constants c_3 and c_4 are left). If the exit cross section has a different orientation, one will first determine particular solutions which are periodic in the $\bar{\xi}$ direction. They will then contain terms $\bar{\eta} \exp(i\omega(\mp\bar{\xi}+\bar{\eta}))$. By linear combinations with the expression (Eq. (322)) one then can generate solutions which are periodic in an arbitrary direction. By imposing conditions at an exit cross section one will always suppress particular solutions except for those of the original flow equations.

The occurrence of terms $\bar{\xi} \exp(i\omega(\mp\bar{\xi}+\bar{\eta}))$ or $\bar{\eta} \exp(i\omega(\mp\bar{\xi}+\bar{\eta}))$ is indicative of the confluence of particular solutions with slightly different values of ω. In the discretized case one will have only incipient confluence, but the awareness of this state of affairs will be helpful in the treatment of results.

The discretized treatment starts with Eq. (317).
Introducing again the expressions (Eq. (300))

$$a_1 = (1 - M^2 \cos^2\theta)$$

$$a_2 = -M^2 \sin\theta \cos\theta$$

$$a_3 = (1 - M^2 \sin^2\theta)$$

one has

$$J = (1/2) \int_\Omega \{ [a_1\phi_x + a_2\phi_y - \psi_y]^2 + [a_2\phi_x + a_3\phi_y + \psi_x]^2 \} dx dx$$

or

$$J = (1/2) \int_\Omega \{ (b_1\phi_x^2 + b_2\phi_y^2 + 2b_3\phi_x\phi_y + 2b_4\phi_x\psi_x + 2b_5\phi_y\psi_x$$

$$+ 2b_6\phi_x\psi_y + 2b_7\phi_y\psi_y + \psi_x^2 + \psi_y^2) dx dx$$

with

$$b_1 = a_1^2 + a_2^2 , \quad b_2 = a_2^2 + a_3^2, \quad b_3 = a_2(a_1 + a_3)$$

$$b_4 = a_2, \quad b_5 = a_3, \quad b_6 = -a_1, \quad b_7 = -a_2$$

Now independent variations of ϕ and ψ are applied. ϕ and ψ are represented by the same shape functions χ_{mn}, chosen as in Section XIII. One obtains by varying ϕ

$$\int_\Omega (b_1\phi_x\chi_{mn,x} + b_2\phi_y\chi_{mn,y} + b_3(\phi_x\chi_{mn,y} + \phi_y\chi_{mn,x}) + b_4\psi_x\chi_{mn,x}$$

(323)

$$+ b_5\psi_x\chi_{mn,y} + b_6\psi_y\chi_{mn,x} + b_7\phi_{mn,y}) dx dy = 0$$

and by varying ψ

$$\int_\Omega (b_4 \phi_x \chi_{mn,x} + b_5 \phi_y \chi_{mn,y} + b_6 \phi_x \chi_{mn,y} + b_7 \phi_y \chi_{mn,y}$$

$$+ \psi_x \chi_{mn,x} + \psi_y \chi_{mn,y})dxdy = 0 \qquad (324)$$

The expressions for the integrals are found in Eqs. (302), (303), and (304). The expressions with ψ are, of course, given by the same formulae. Next, one writes in analogy to Eq. (305)

$$\begin{pmatrix} \phi_{mn} \\ \psi_{mn} \end{pmatrix} = \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \end{pmatrix} \exp(im\omega_3 \Delta x + in\omega_4 \Delta y) \qquad (325)$$

where $\tilde{\phi}$ and $\tilde{\psi}$ are constants. As in Eq. (292)

$$\omega_3 = \omega_1 \cos \nu - \omega_2 \sin \nu$$

$$\omega_4 = \omega_1 \sin \nu + \omega_2 \cos \nu \qquad (326)$$

where ω_2 is real.

Substituting the expressions for the integrals given in Eqs. (306), (307), and (308) into Eqs. (323) and (324) one obtains two homogeneous equations for $\tilde{\phi}$ and $\tilde{\psi}$ with coefficients which are expressed in terms of given quantities M^2, Δx, Δy and in terms of ω_3 and ω_4. The latter expressions are expressed by Eqs. (326) in terms of ω_1 and ω_2, where ω_2 can be considered as known.

The determinant of this two-by-two system for $\tilde{\phi}$ and $\tilde{\psi}$ must vanish. This gives ultimately an equation for ω_1 which must be solved numerically. One will proceed as follows. For given ω_2 and assumed ω_1 one determines ω_3 and ω_4. To evaluate the integrals one sets

$$r_1 = \exp(i\omega_4 \Delta y)$$

$$r_2 = \exp(i\omega_3 \Delta x) \qquad (327)$$

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Except for some common factors, one obtains from Eqs. (306), (307), and (308) expressions

$$I_1 = (\Delta y/\Delta x)[(2/3) + (1/6)(r_1^{-1} + r_1)][2 - r_2^{-1} - r_2]$$

$$I_2 = (1/4)[-r_1^{-1} + r_1][r_2^{-1} - r_2] \qquad (328)$$

$$I_3 = (\Delta x/\Delta y)[2 - r_1^{-1} - r_1][(2/3) + (1/6)(r_2^{-1} + r_2)]$$

Substitution into Eqs. (323) and (324) then gives the following system of equations

$$\begin{pmatrix} (b_1 I_1 + b_2 I_3 + 2b_3 I_2) & (b_4 I_1 + (b_5+b_6)I_2 + b_7(I_3) \\ b_4 I_1 + (b_5+b_6)I_2 + b_7 I_3) & (I_1+I_3) \end{pmatrix} \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \end{pmatrix} = 0 \qquad (329)$$

The postulate that the determinant vanishes gives an equation from which the assumed quantity ω_1 is determined. The roots so found are noted by $\omega_{1,j}$. (The pertinent vectors will be denoted by $(\tilde{\phi}_j, \tilde{\psi}_j)^\dagger$ or, for brevity, by Ω_j.) Regarding the roots we make the following observation. Assume that the discretized problem has been solved for a fixed set of basic parameters. Then one has for some real ω_2 a complex value ω_1 for which the determinant of Eqs. (329) vanishes. Assume now that ω_1 is replaced by its conjugate comple ω_1^*. Then by Eqs. (236) ω_3 and ω_4 are replaced by ω_3^* and ω_4^*, and by Eq. (327) r_1 and r_2 are replaced by $\exp(i\omega_4^*\Delta y)$ and $\exp(i\omega_3^*\Delta x)$. Writing $\exp(i\omega_4^*\Delta y)$ in the form $\exp(i(\mathrm{Re}\ \omega_4 - \mathrm{Im}\omega_4)\Delta y)$ one recognizes that r_1 and r_2 are replaced by r_1^{-1*} and r_2^{-1*}. Because of the special form in which r_1 and r_2 occur in the expressions I_1, I_2, and I_3, Eqs. (328), these expressions are then replaced by I_1^*, I_2^*, and I_3^*. Finally, the determinant of Eqs. (329) is replaced by its conjugate complex; if it vanishes in the original form, it will also vanish if ω_1 is replaced by ω_1^*. It follows that to each value of r_1 or r_2 which solves the problem, there exists a second "root" r_1^{-1*} and r_2^{-1*}, r_1^{-1*} has the same argument as r_1, but its modulus is replaced by its reciprocal.

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The values of $\tilde{\phi}$ and $\tilde{\psi}$, for which Eqs. (329) are solved are also replaced by their reciprocals.

For $\nu = 0$ (the only case investigated numerically in this report) $\omega_4 = \omega_2$ (see Eq. (326)) is known, and $\omega_3 = \omega_1$; only r_2, now denoted by r, is then unknown. Then the expressions (Eqs. (328)) assume the form

$$I_1 = c_1(-r^{-1} + 2 - r)$$
$$I_2 = ic_3(r^{-1} - r) \qquad\qquad (330)$$
$$I_3 = c_2(r^{-1} + 4 + r)$$

with

$$c_1 = (\Delta y/\Delta x)[(2/3) + (1/3)\cos(\omega_2\Delta y)]$$
$$c_2 = (1/3)(\Delta x/\Delta y)[1 - \cos(\omega_2\Delta y) \qquad\qquad (331)$$
$$c_3 = (1/2)\sin \omega_2\Delta y$$

Substituting this into Eqs. (323) and (324), one obtains

$$[d_1(-r^{-1} + 2 - r) + d_2(r^{-1} + 4 + r) + d_3 i(r^{-1} - r)]\tilde{\phi}$$
$$\qquad\qquad (332)$$
$$+ [d_4(-r^{-1} + 2 - r) + d_5(r^{-1} + 4 + r) + d_6 i(r^{-1} - r)]\tilde{\psi} = 0$$

and a second equation with the d_j's replaced by e_j's. Here

$$d_1 = b_1c_1, \ d_2 = b_2c_2, \ d_3 = 2b_3c_3, \ d_4 = b_4c_1,$$
$$\qquad\qquad (333)$$
$$d_5 = b_1c_2, \ d_6 = (b_5 + b_6)c_3$$

$$e_1 = b_4c_1, \ e_2 = b_7c_2, \ e_3 = (b_5 + b_6)c_3,$$
$$\qquad\qquad (334)$$
$$e_4 = c_1, \ e_5 = c_2, \ e_6 = 0$$

Ordering the terms with respect to powers of r, one obtains

$$
\begin{pmatrix} (f_1 r^{-1} + f_2 + f_3 r) & (f_4 r^{-1} + f_5 + f_6 r) \\ (g_1 r^{-1} + g_2 + g_3 r) & (g_4 r^{-1} + g_5 + g_6 r) \end{pmatrix} \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \end{pmatrix} = 0
$$

with

$$
\begin{aligned}
f_1 &= -d_1 + d_2 + id_3 \quad , \quad f_2 = 2d_1 + 4d_2 \\
f_3 &= -d_1 + d_2 - id_3 \quad , \quad f_4 = -d_4 + d_5 + id_6 \\
f_5 &= 2d_4 + 4d_5 \qquad\qquad , \quad f_6 = -d_4 + d_5 - id_6
\end{aligned}
\tag{335}
$$

The expressions for the coefficients g_6 arise by replacing the f_j's by g_j's and the d_k's by e_k's. The requirement that the determinant vanish gives a fourth degree equation in r.

$$
A_4 r^4 + A_3 r^3 + A_2 r^2 + A_1 r + A_0 = 0
\tag{336}
$$

Here

$$
\begin{aligned}
A_4 &= f_3 g_6 - g_3 f_6 \\
A_3 &= (f_3 g_5 + f_2 g_6) - (g_3 f_5 + g_2 f_6) \\
A_2 &= (f_3 g_4 + f_2 g_5 + f_1 g_6) - (g_3 f_4 + g_2 f_5 + g_1 f_6) \\
A_1 &= (f_2 g_4 + f_1 g_5) - (g_2 f_4 + g_1 f_5) \\
A_4 &= f_1 g_4 - f_4 g_1
\end{aligned}
\tag{337}
$$

In the computations one must evaluate the a_i's, b_i's...g_i's , and the coefficients of the last equation. Then a root finding routine can be applied. The four roots obtained are denoted by r_j and the corresponding vectors $(\tilde{\phi}, \tilde{\psi})^+$ by Ω_j.

The general solution is then given by

$$
\Omega_{mn} = \exp(in\omega \Delta y) \sum_{i=1}^{4} \alpha_j \Omega_j r_j^m
\tag{338}
$$

For $\theta = 0$ one expects to find some further simplification, because of the ensuing symmetry of the solution with respect to the x-axis. A closer examination shows, indeed, that one can split the fourth degree equation into two second degree equations. First one evaluates an intermediate variable, $v = (r^{-1} + r)$. Having found values of v, one can compute r. There is little point in using this simplification unless the entire computation is restricted to the special case $\theta = 0$.

The four solutions of the discretized Euler equations give approximation to the two solutions of the original flow problem. In the discussions of the Euler equation carried out at the beginning of this section, we found two particular solutions which satisfy the linearized flow equations and two other particular solutions, arising by confluence, which fail to solve the original flow equations. The latter are suppressed as one satisfies the boundary conditions for the Euler equations at the exit cross section. (These boundary conditions express the requirement that there the residuals be zero.) Such boundary conditions will, of course, arise also in the discretized treatment. After they have been taken into account, the number of particular solutions available to satisfy conditions at the entrance cross section is reduced to two.

For $v = 0$ the exit cross section coincides with element boundaries. The conditions at the exit cross section then arise automatically by varying ϕ and ψ at the exit; the procedure arises entirely from the concept of residual minimization. For $v \neq 0$ the corresponding steps can be caried cut only if one introduces irregular elements for which part of the boundary forms the exit cross section. Unfortunately such a procedure is not compatible with the present method of analysis which presupposes that all elements have the same size and shape. The expressions Eqs. (325) give ϕ and ψ only at the corners of the elements, while these quantities are defined in the interior of the elements by means of the shape functions. For the purpose of the analysis of the case $v \neq 0$, we pretend that Eq. (325) is valid everywhere in the exit section, i.e., we assume that there

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$$\begin{pmatrix} \phi(x,y) \\ \psi(x,y) \end{pmatrix} = \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \end{pmatrix} \exp(i\omega_3 x + i\omega_4 y) \qquad (339)$$

where ω_3 and ω_4 are determined by the procedure for the discretized equations described above. With the assumption Eq. (339) one can use the boundary conditions that arise from the nondiscretized Eq. (317). For this purpose we introduce the residuals

$$R_1 = (1 - M^2 \cos^2\theta)\phi_x - M^2 \sin\theta \cos\theta \; \phi_y - \psi_y$$

$$\qquad (340)$$

$$R_2 = -M^2 \sin\theta \cos\theta \; \phi_x + (1 - M^2 \sin^2\theta)\phi_y + \psi_x$$

By definition the exit cross section is formed by a line $\xi_1 = $ const. We had

$$x = \xi_1 \cos\nu - \eta_1 \sin\nu$$

$$\qquad (341)$$

$$y = \xi_1 \sin\nu + \eta_1 \cos\nu$$

Therefore along the exit cross section $(\xi_1 = $ const$)$

$$\frac{dx}{d\eta_1} = -\sin\nu; \qquad \frac{dy}{d\eta_1} = \cos\nu \qquad (342)$$

Using these equations one obtains from Eq. (317) as boundary terms which arise by varying ϕ and ψ in the end section

$$M^{(1)} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = 0 \qquad (343)$$

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where

$$M^{(1)} =$$

$$\begin{pmatrix} (1-M^2\cos^2\theta)\cos\nu-M^2\sin\theta\cos\theta\sin\nu & -M^2\sin\theta\cos\theta\cos\nu+(1-M^2\sin^2\theta)\sin\nu \\ -\sin\nu & \cos\nu \end{pmatrix}$$

(344)

Eq. (333) has arisen from an expression in terms of ξ_1 and η_1

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \end{pmatrix} \exp(i\omega_1\xi_1 + i\omega_2\eta_1)$$

The computation is carried out in such a manner that ω_2 is real and the same for all roots. One obtains four different expressions differentiated by a subscript j. We choose the origin of the coordinate systems in the exit section, then $\xi_1 = 0$. The residuals are evaluated at the exit cross section separately for the four solutions from Eq. (340)

$$\begin{pmatrix} R_{1,j} \\ R_{2,j} \end{pmatrix} = i \exp(i\omega_1\eta_1)M_j^{(2)}\begin{pmatrix} \tilde{\phi}_j \\ \tilde{\psi}_j \end{pmatrix}$$

(345)

with

$$M_j^{(2)} = \begin{pmatrix} [(1 - M^2\cos^2\theta)\omega_{3,j} - M^2\sin\theta\cos\theta\,\omega_{4,j}] & -\omega_{4,j} \\ [-M^2\sin\theta\cos\theta\,\omega_{3,j} + (1 - M^2\sin^2\theta)\omega_{4,j}] & \omega_{3,j} \end{pmatrix}$$

(346)

If one sets

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sum_j \alpha_j \begin{pmatrix} \tilde{\phi}_j \\ \tilde{\psi}_j \end{pmatrix} \exp(i\omega_{3,j}x + i\omega_{4,j}y)$$

(347)

one obtains, from the requirement Eq. (343)

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$$A_j \, \alpha_j = 0$$

$$B_j \, \alpha_j = 0 \tag{348}$$

with

$$
\begin{aligned}
A_j &= M^{(1)} \ R_{1,j} \\
B_j &\qquad\quad R_{2,j}
\end{aligned} \tag{349}
$$

These are two equations for the four unknown coefficient α_j. The computation evaluates $M^{(1)}$, $M_j^{(2)}$, $(R_{1,j}, R_{2,j})$, and (A_j, B_j). The procedures just described can be carried out also for $\nu = 0$. But, as was mentioned above, for this case one can also use the conditions which arise from minimizing the discretized functional J and avoid the assumption Eq. (339). Varying ϕ and ψ in the exit cross section on obtains Eqs. (323) and (324) (as everywhere else in the flow field), but the integrals occurring in these equations are expressed by the shape parameters in a different form because only elements lying upstream of the exit cross section will give contributions.

Denote by me the value of m for the exit section. Then one has

$$\int_\Omega \phi_x \, x_{me,n,x} \, dx\,dy =$$
$$\Delta x^{-1} \Delta y [(2/3)(\phi_{me,n} - \phi_{me-1,n}) + (1/6)(\phi_{me,n-1} - \phi_{me-1,n-1})$$
$$+ (1/6)(\phi_{me,n+1} - \phi_{me-1,n+1})]$$

$$\int_\Omega \phi_x \, x_{me,n,y} \, dx\,dy = (1/4)[\phi_{me,n-1} - \phi_{me-1,n-1} - \phi_{me,n+1} + \phi_{me-1,n+1}]$$

$$\tag{350}$$

$$\int_\Omega \phi_y \, x_{me,n,x} \, dx\,dy = (1/4)[-\phi_{me,n-1} + \phi_{me,n+1} + \phi_{me,n+1} - \phi_{me-1,n-1}]$$

$$\int_\Omega \phi_y \, x_{me,n,y} \, dx\,dy = \Delta x \Delta y^{-1}[-(1/3)\phi_{me,n-1} + (2/3)\phi_{me,n} - (1/3)\phi_{me,n+1}$$
$$- (1/6)\phi_{me-1,n-1} + (1/3)\phi_{me-1,n} - (1/6)\phi_{me-1,n+1}]$$

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Because the present development holds only for $\nu = 0$, we set then

$$\phi_{mn} = r^m \exp(in\omega_2 \Delta y) \qquad (351)$$

With the constant c_1, c_2, and c_3 defined in Eq. (331), one has

$$\exp(-i(n\omega_2\Delta y + me\omega_1\Delta x)) \int_\Omega \phi_x x_{me,n,x}\,dxdy = c_1(\phi_{me} - \phi_{me-1})$$

$$\exp(-i(n\omega_2\Delta y + me\omega_1\Delta x)) \int_\Omega \phi_y x_{me,n,y}\,dxdy = c_2(2\phi_{me} + \phi_{me-1})$$

$$\exp(-i(n\omega_2\Delta y + me\omega_1\Delta x)) \int_\Omega \phi_x x_{me,n,y}\,dxdy = ic_3(-\phi_{me} + \phi_{me-1})$$

$$\exp(-i(n\omega_2\Delta y + me\omega_1\Delta x)) \int_\Omega \phi_y x_{me,n,x}\,dxdy = ic_3(\phi_{me} + \phi_{me-1})$$

Then by substitution into Eqs. (323) and (324) with coefficients d_i and e_i defined in Eqs. (333) and (334)

$$d_1(\phi_{me} - \phi_{me-1}) + d_2(2\phi_{me} + \phi_{me-1}) + id_3\phi_{me-1} + d_4(\psi_{me} - \psi_{me-1})$$
$$+ d_5(2\psi_{ne} + \psi_{me-1}) + b_5 c_3 i(-\psi_{me} + \psi_{me-1}) + b_6 c_3 i(\psi_m + \psi_{m-1}) = 0$$

and

$$e_1(\phi_{me} - \phi_{me-1}) + e_2(2\phi_{me} + \phi_{me-1}) + b_5 c_3 i(\phi_{me} + \phi_{me-1})$$
$$+ b_6 c_3 i(-\phi_{me} + \phi_{me-1}) + e_4(\psi_{me} - \psi_{me-1}) + e_5(2\psi_{me} + \psi_{me-1}) = 0$$

Ordering according to subscripts one obtains

$$h_1\phi_{me} + h_2\phi_{me-1} + h_3\psi_{me} + h_4\psi_{me-1} = 0$$

$$\qquad (352)$$

$$k_1\phi_{me} + k_2\phi_{me-1} + k_3\psi_{me} + k_4\psi_{me-1} = 0$$

with

$$h_1 = d_1 + d_2$$

$$h_2 = -d_1 + d_2 + id_3$$

$$h_3 = d_4 + 2d_5 + ic_3(-b_5 + b_6)$$

$$h_4 = -d_4 + d_5 + ic_3(b_5 + b_6)$$

$$k_1 = e_1 + 2e_2 + ic_3(b_5 - b_6)$$

$$k_2 = -e_1 + e_2 + ic_3(b_5 + b_6)$$

$$k_3 = e_4 + 2e_5$$

$$k_4 = -e_4 + e_5$$

(353)

Substituting Eq. (338) into Eq. (352) one obtains equations for the coefficients α_i

$$\sum_{j=1}^{4} A_j \alpha_j = 0$$

$$\sum_{j=1}^{4} B_j \alpha_j = 0$$

(354)

where

$$A_j = (h_1 + \bar{r}_j^1 h_2)\tilde{\phi}_j + (h_3 + \bar{r}_j^1 h_4)\tilde{\psi}_j$$

$$B_j = (k_1 + \bar{r}_j^1 k_2)\tilde{\phi}_j + (k_3 + \bar{r}_j^1 k_4)\tilde{\psi}_j$$

(355)

Eqs. (354) replace Eqs. (348).

The limiting case in which the exit cross section (ξ_1 = const) coincides with a Mach wave has some special properties, at least in the nondiscretized equations. In this limit one has

$$\nu = \mp(\pi/2) + (\theta \pm \alpha)$$

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therefore

$$\cos \nu = \pm \sin(\theta \pm \alpha)$$

(356)

$$\sin \nu = \mp \cos(\theta \pm \alpha)$$

For simplicity, only the upper sign is included in the following discussion. Results for the lower sign are immediately obtained by changing the sign of α.

We shall see, that in this limiting case the two equations (Eq. (347)) are linearly dependent. One, therefore, obtains only one condition for R_1 and R_2 instead of two. By substitution of Eqs. (356) one obtains from the first row of the matrix Eq. (344)

$$[-(\sin^2\alpha - \cos^2\theta)\sin(\theta+\alpha) - \sin\theta\,\cos\theta\,\cos(\theta+\alpha)]R_1$$

(357)

$$+ [\sin\theta\,\cos\theta\,\sin(\theta+\alpha) + (\sin^2\alpha - \sin^2\theta)\cos(\theta+\alpha)]R_2 = 0$$

The evaluation of the coefficients of R_1 and R_2 uses the following identities

$$\sin^2\alpha - \sin^2\theta = \sin(\alpha+\theta)\sin(\alpha-\theta)$$

(358)

$$\sin\alpha\,\cos\alpha - \sin\theta\,\cos\theta = \sin(\alpha-\theta)\cos(\alpha+\theta)$$

(359)

and consequently

$$\sin^2\alpha - \cos^2\theta = -\cos(\alpha+\theta)\cos(\alpha-\theta)$$

(360)

and

$$\sin\alpha\,\cos\alpha + \sin\theta\,\cos\theta = \sin(\alpha+\theta)\cos(\alpha-\theta)$$

(361)

The relations can be verified by applying familiar trigonometric identities to the right-hand sides. They can be "understood" if one observes that the left- and the right-hand sides are, for

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fixed θ, analytic functions of α with the same zeros. In Eq. (358) the zeros are, for instance

$$\alpha = \theta + n_1 \pi \text{ and } \alpha = -\theta + n_1 \pi,$$

where n_1 is any integer.

One obtains for the coefficient of R_1 in Eq. (357) by applying Eqs. (360) and (361)

$$-(\sin^2 \alpha - \cos^2 \theta)\sin(\theta + \alpha) - \sin \theta \cos \theta \cos(\theta + \alpha)$$

$$= \cos(\alpha + \theta)\cos(\alpha - \theta)\sin(\theta + \alpha) - \sin \theta \cos \theta \cos(\theta + \alpha)$$

$$= \cos(\theta + \alpha)[\cos(\alpha - \theta)\sin(\alpha + \theta) - \sin \theta \cos \theta]$$

$$= \sin \alpha \cos \alpha \cos(\theta + \alpha)$$

The coefficient of R_2 in Eq. (357) is obtained by an analogous procedure; one obtains

$$\sin \alpha \cos \alpha \sin(\theta + \alpha)$$

Accordingly, one finds from Eq. (357)

$$R_1 + tg(\theta + \alpha)R_2 = 0 \qquad\qquad (362)$$

The same result is found (without further computations) from the second row of Eq. (344).

One might surmise that Eq. (362) is related to the characteristic condition for the basic flow equations, which arise by setting the expression in Eq. (340) equal to zero. The characteristic conditions are obtained by forming a linear combination of these two equations in such a manner that only derivatives in the direction of the characteristics are encountered. In the present example the characteristics satisfy

$$\frac{dy}{dx} = tg(\theta + \alpha)$$

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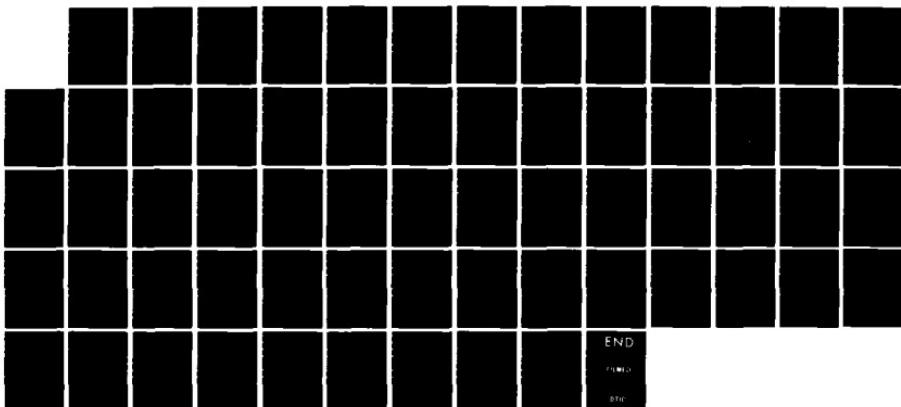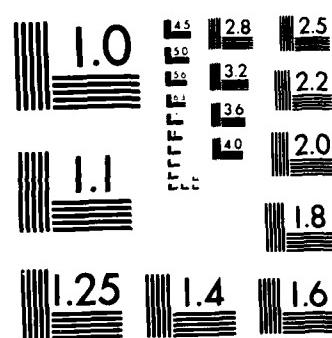
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The linear combination to be formed is recognized immediately if one considers the derivatives of ψ. To obtain $\psi_x + tg(\theta+\alpha)\psi_y$ one must form

$$R_2 - tg(\theta+\alpha)R_1$$

The characteristic condition for the ϕ, ψ representation is, therefore,

$$R_1 - \cot(\theta+\alpha)R_2 = 0 \qquad (363)$$

One is led to a correct interpretation of Eq. (362) by the observation that Eq. (343) represent initial conditions at the exit cross section for the Euler equations interpreted as differential equations for the residuals R_1 and R_2. The Euler equations (Eqs. (318) and (319)) rewritten in terms of R_1 and R_2 (defined in Eq. (340)) read after multiplication by $M^{-2} = \sin^2\alpha$

$$(\sin^2\alpha - \cos^2\theta)(\partial R_1/\partial x) - \sin\theta\cos\theta(\partial R_1/\partial y)$$
$$\qquad (364)$$
$$- \sin\theta\cos\theta(\partial R_2/\partial x) + (\sin^2\alpha - \sin^2\theta)(\partial R_2/\partial y) = 0$$

and

$$- (\partial R_1/\partial y) \qquad + (\partial R_2/\partial x) = 0$$

The characteristic conditions for this system of equations are obtained by multiplying the first equation by $(\sin^2\alpha - \cos^2\theta)^{-1}$, the second by $\sin\alpha\cos\alpha(\sin^2\alpha - \cos^2\theta)^{-1}$, and by adding the results. One obtains

$$\left[\frac{\partial R_1}{\partial x} + \frac{-\sin\theta\cos\theta - \sin\alpha\cos\alpha}{\sin^2\alpha - \cos^2\theta}\frac{\partial R_1}{\partial y}\right]$$

$$+ \frac{-\sin\theta\cos\theta + \sin\alpha\cos\alpha}{\sin^2\alpha - \cos^2\theta}\left[\frac{\partial R_2}{\partial x} + \frac{\sin^2\alpha - \sin^2\theta}{-\sin\theta\cos\theta + \sin\alpha\cos\alpha}\frac{\partial R_2}{\partial y}\right] = 0$$

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Hence with Eqs. (358) through (362)

$$(\partial R_1/\partial x) + tg((\theta+\alpha)(\partial R_1/\partial y) + tg(\theta-\alpha)(\partial /R_2/\partial x)$$
$$+ tg(\theta+\alpha)(\partial R_2/\partial) = 0$$

Hence

$$R_1 + tg(\theta-\alpha)R_2 = const \qquad (365)$$

along a line with the slope $(dy/dx) = tg(\theta+\alpha)$, (a left-going characteristic). For the right-going characteristic one must replace α by $-\alpha$.

$$R_1 + tg(\theta+\alpha)R_2 = const, \qquad (366)$$

there

$$dy/dx = tg(\theta-\alpha)$$

The constants in Eqs. (365) and (366) may be different for each characteristic.

One then arrives at the following interpretation of Eq. (362). We assumed that the exit cross section is given by a left-going characteristic (Fig. 13). Eq. (362) then gives zero for the value of the constant in Eq. (366), that means for the right-going characteristics which pass through the exit cross section. This information together with the homogeneous boundary condition for R_1 and R_2 which are given at the side walls (initially at the upper wall) cause R_1 and R_2 to be zero; first along the exit cross section and afterwards in the construction by a method of characteristics all through the flow field. We observed at the beginning of this discussion that the two equations (Eq. (347)) yield in the present limit only one condition for R_1 and R_2, this is Eq. (362). No further condition is needed because one assumes that the Euler equation of the variational problem, here interpreted as the system of partial differential equations for R_1

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and R_2, Eq. (364) are satisfied. For the left-going charac-
teristic this system reduces to Eq. (365). The constant in this
equation is zero, because of Eq. (362) in combination with the
homogeneous condition at the side wall.

After this digression, we return to the application of the
conditions in the exit cross section, Eqs. (348) or Eqs. (355).
We want to define solutions of the Euler equations (now in
discretized form) which make the residuals zero at the end cross
section and which are suitable for a comparison with exact
solutions of the original flow equations. The simplest analytical
solutions have the form of single waves. In the ξ, η system (the
one oriented with respect to the velocity vector of the basic
flow) they are given by

$$\phi = \exp(i\omega_2(tg\alpha\ \xi + \eta) \text{ and } \phi \exp(i\omega_2(-tg\alpha\ \xi + \eta)$$

Similarly, equations hold for ψ. At least for ω_2 small, the four
particular solutions of the discretized Euler equations which
appear in Eq. (347) can be ordered into two pairs (recognizable by
their closeness to the values of ω_1 taken from the exact solution
for single waves). Under conditions where the approximation to
the exact solutions is only poor (larger values of ω, and
proximity to certain limiting cases) one can carry out the
ordering by continuity with respect to established cases.

Eqs. (348) or (355) are two homogeneous equations with four
unknowns a_i. Only the ratio of the unknowns matters. One cannot
expect that these relations can be satisfied if one uses only the
two expressions which are "close" to the exact particular
solutions for a single wave. It has been observed that the values
of r_1 and r_2 (Eq. (327)) belonging to such a pair have the same
phase and moduli which are reciprocals of each other. In each
pair one of the approximating particular solutions therefore
decreases and the other increases as one moves from the exit cross
section towards the interior in the x direction. To satisfy
Eqs. (348) or (355) we now use the pair of particular solutions
which are approximations to the single wave which we want to study

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(in our examples these are right-going waves) and the one particular solution pertaining to the approximations for the other wave (here left-going) which decreases as one moves from the exit cross section toward the interior. The contribution of this latter particular solution will then become smaller and smaller in comparison to the approximation for the right-going wave. So ultimately only the latter particular solutions are encountered. In the cases studied numerically the contribution of the undesired waves (left-going) is very small (much less than 1 percent) even at the exit cross section.

For a fixed set of parameters M^2, $\omega_2 \Delta y$, $\Delta x/\Delta y$, and θ the numerical procedure evaluates the constants a_i (Eq. (300), b_i (formulae preceding Eq. 323), c_i, d_i, e_i, f_i, g_i (Eqs. (330), (331), (333), (334), (335)) and the coefficients A_i (Eq. (337)) of the equation of fourth degree (Eq. (336)). (For the particular case $\nu = 0$ treated here, r replaced r_2.) The values of r and the pertinent vectors ($\tilde{\phi}, \tilde{\psi}$) appear in pairs, in which the arguments of the r's are the same while the moduli are reciprocals of each other. By comparison of the argument of r with that of exact solutions they are identified as approximation to either left- or right-going waves. One then evaluates the constants h_i and k_i (Eqs. (354)) and finally, by Eq. (356), the coefficents A_j and B_j which occur in Eq. (355).

Now one chooses in Eq. (355) the one α_j equal to zero which pertains to the approximation of the undesired (left-going) wave that increases as one proceeds from the exit cross section toward the interior and computes the ratio of the three other coefficients α_j. It is in this computation where we found that the coefficient of the other undesired particular solution (here of a left-going wave) is very small. The solutions are normalized so that the value of ϕ at the exit cross section is one.

NUMERICAL RESULTS FOR THE LINEARIZED SUPERSONIC FLOW

In a specific situation the values of M^2, θ, and $\Delta x/\Delta y$ are fixed. The values of $\omega_2 \Delta y$ may range from 0 to π. It is then possible to evaluate for the particular solutions belonging to these values of $\omega_2 \Delta y$ the approximations to right-going and left-going waves. Two questions arise: whether the procedure is stable for all possible values of $\omega_2 \Delta y$ and up to which values of $\omega_2 \Delta y$ one obtains a reasonable accuracy.

The results are summarized in a number of tables. We have chosen three Mach numbers ($M^2 = 1.5$, $M^2 = 2$, and $M^2 = 2.5$). In the tables we have varied $p_2 = \omega_2 \Delta y$ and rather clear cut results are obtained if one chooses $\Delta x/\Delta y$ in such a manner that $p_1 = \omega_1 \Delta x$ remains the same for the exact solutions by right-going waves as one varies θ at a fixed value of p_2. (The expressions $p_1/(2\pi)$ and $p_2/(2\pi)$ give the number of mesh points per full wave in the x and y directions.) If one keeps p_1 and p_2 constant as θ varies, then one has the possibility of representing the particular solutions which arise for different values of θ with about the same accuracy. However, this representation holds only for either right- or left-going waves. For these cases one has

$$\frac{\Delta x}{\Delta y} = \frac{p_1}{p_2} \cot(\theta \overline{+} \alpha)$$

(The upper and lower signs for right- and left-going waves, respectively.) In practice $\Delta x/\Delta y$ is fixed. Therefore a second set of tables has been prepared with

$$\frac{\Delta x}{\Delta y} = \gamma \cos t \, \alpha$$

Then γ is the Courant number of the net of grid points for $\theta = 0$. Actually the evaluation has been carried out only for $\gamma = 1$. The tables contain a column for

$$p_1/p_2 = \gamma \cos \alpha \, tg(\theta-\alpha)$$

which allows one to determine the number of grid points per wave length in the x direction which may vary dramatically.

In the discussions of Section XIII we have seen that with the use of the Bristeau method, one obtains particular solutions which retain their amplitude but may have an error in phase (the discretized scheme is dispersive). This is correct up to a certain value of $\Delta x/\Delta y$ from there on one of the particular solutions will increase as one moves downstream, that is the method is unstable. The tables show the phase error after a full wave. One must decide which wave lengths must be represented with accuracy and adjust them ratio $\Delta x/\Delta y$ accordingly. Cases which are unstable are left blank in the table for the phase errors.

For practical purposes the set of tables in which $\Delta x/\Delta y$ is fixed is, of course, more important. It is disconcerting that the result depends rather strongly upon the alignment of the mesh with the velocity vector (this is expressed by the angle θ). We started from the idea that the methods of residual minimization guarantee convergence. They do, but for a given mesh they may be rather inaccurate in the supersonic region.

For the ϕ,ψ method one may have a phase error, as well as damping in any case. For each particular solution for a single wave of the nondiscretized equation there are two approximations, one which decreases and the other which increases in the x direction. The boundary conditions in the exit cross section have the effect that the contribution of these two parts are of nearly equal magnitude in the exit cross section. In the interior of the flow field the one approximation will prevail which decreases as one moves toward the exit section. This has the consequence that all solutions are stable. Two tables are given for each case, one for the phase error over one full wave in the x direction and the other one for the amplitude of the resulting approximation of one full wave (in the x direction and of the exit section).

All tables are extended to values of θ close to the limit where one would have a well-posed problem. Because we have chosen ν = 0 (upstream and downstream boundary given by lines x = const) these limits are $-(\frac{\pi}{2} - \alpha) < \theta < + (\frac{\pi}{2} - \alpha)$. For $M^2 > 2$, ($\alpha < \frac{\pi}{4}$), cases may arise within these limits for which θ = α and the x-axis coincides with a right-going wave. The wave length in the x direction is then infinite. In the tables this is the point where Δx/Δy changes sign. This, however, does not lead to any spectacular changes in the vicinity of this point.

The results show that in a supersonic region the method frequently fails to give results of high accuracy, especially for a nonaligned grid, (even while the regions of influence are observed properly). Similar phenomena will be encountered in three-dimensional cases. Other discretizations (for instance by triangular elements) applied with or without residual minimization may show similar phenomena. Because of lack of time the investigations have not been extended in this direction. Even though the present setting gives approximations to the particular solutions which in the φ,ψ representation are damped, it is compatible with rarefaction shocks; therefore, it will be necessary to introduce measures to suppress them.

APPENDIX A
MATRIX MODIFICATION

Assume that we have a problem

$$(M_1 + M_2)y = z \qquad (A.1)$$

in which M_1 is a nondegenerate matrix of dimension d by d and the modifying matrix M_2 can be partitioned as follows.

$$M_2 = \begin{pmatrix} 0 & | & \overline{M}_2 & | & 0 \end{pmatrix}$$

The dimension of $\overline{M}_2$ is d by d_2. The following approach leaves the structure of the matrix M_1 intact; a block tri-diagonal matrix for instance, will not be altered. It follows from Eq. (A.1)

$$(I + M_1^{-1}M_2)\vec{y} = M_1^{-1}\vec{z} \qquad (A.2)$$

Notice that $M_1^{-1}M_2$ has the same partitioning as M_2. We partition y and $\vec{u} = M_1^{-1}z$ in the manner corresponding to the partitioning of M_2. The subvectors are $\vec{y}_1$, $\vec{y}_2$, $\vec{y}_3$ and for $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$. The matrix $(M^{-1}M_2)$ is partitioned row wise in the same manner as y and u and the portion corresponding to y_2 or u_2 is denoted by $(M^{-1}M_2)_2$. Restricting oneself to the component with subscript 2 one has

$$(I + (M^{-1}M_2)_2)\vec{y}_2 = \vec{u}_2 \qquad (A.3)$$

where I is the identity matrix of dimension d_2. Then by substitution into Eq. (A.2)

$$\vec{y} = \vec{u} - M_1^{-1}M_2\vec{y}_2 \qquad (A.4)$$

The determination of $M_1^{-1}z$ requires the solution of one inhomogeneous equation with the matrix M_1 (of dimension d by d). This work must be done even if $M_2 \equiv 0$. Let d_2 be the width of the matrix $\bar{M}_2$. The determination of $M_1^{-1}\bar{M}_2$ required the solution of d_2 inhomogeneous equations again with matrix M_1. The determination of $\vec{y}_2$ requires the inhomogeneous Eq. (A.3), with a matrix of dimension d_2 by d_2. Finally, one must premultiply $\vec{y}_2$ by the matrix $M_1^{-1}M_2$ which has already been evaluated.

APPENDIX B
DENSITY MODIFICATION

The concept of density modification seems to contradict a requirement for the existence of a potential flow, namely that the pressure be a function of only the density. One will, however, remember that a potential flow is compatible with the presence of sources within the flow field (see the remark following Eq. (211)). One is therefore allowed to set

$$\text{div}(\rho \vec{w}) = q \tag{B.1}$$

(q is the source strength). We try to interpret density modification as the presences of sources. For instance, we ask whether Eq. (B.1) can be written in the form

$$\text{div}(\tilde{\rho} \vec{w}) = 0 \tag{B.2}$$

where $\tilde{\rho}$ is a modified density (while the velocity distribution remains the same). Using familiar vector relations one obtains

$$\text{div } \rho \vec{w} = \rho \text{ div } \vec{w} + \text{grad } \rho \cdot \vec{w} = q$$

and

$$\text{div } \tilde{\rho} \vec{w} = \tilde{\rho} \text{ div } \vec{w} + \text{grad } \tilde{\rho} \cdot \vec{w} = 0$$

Hence

$$(\rho - \tilde{\rho}) \text{div } \vec{w} + \text{grad}(\rho - \tilde{\rho}) \cdot \vec{w} = q$$

If the fields $\vec{w}$ and q are known, then, since by Bernoulli's equation ρ is a function of $\vec{w}$, also ρ is known. One then can determine the modification of ρ given by $(\rho - \tilde{\rho})$ by an integration along the streamlines. One can indeed find for each source distribution q a density field $\tilde{\rho}$, so that Eq. (B.2) is satisfied.

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Consider a region Ω of the flow field with boundary $\partial\Omega$. Outside of k and at $\partial\Omega$ $\rho-\tilde{\rho}$ is zero. Let $d\tau$ be the volume element. Then one has

$$\int_{\Omega} q\, d\tau = \int_{\Omega} div(\rho\vec{w})\, d\tau = \int_{\Omega} div((\rho-\tilde{\rho})\vec{w})\, d\tau = \int_{\partial\Omega} (\rho-\tilde{\rho})\vec{w} \cdot \vec{e}_n\, do$$

The fact that $(\rho-\tilde{\rho})$ is zero outside of Ω (and at $\partial\Omega$) then guarantees conservation of mass for the total region Ω. This is of importance if one applies density modification in a region including a shock.
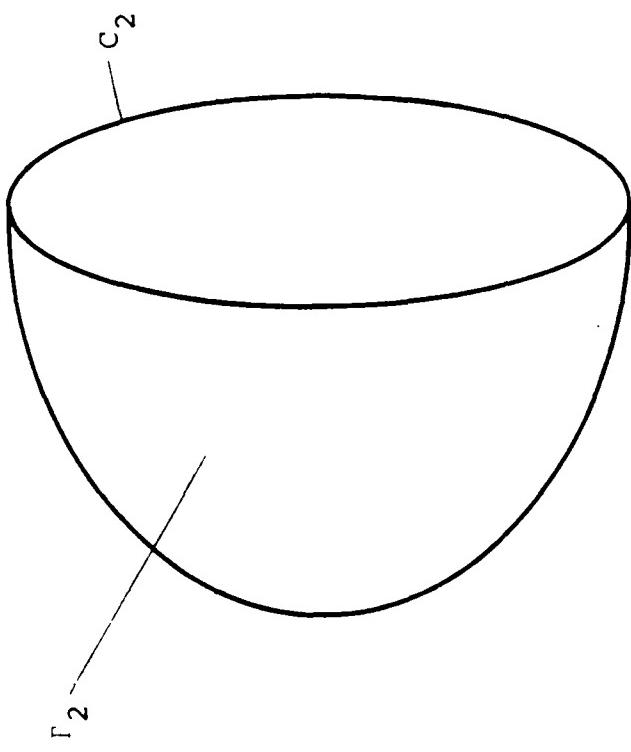
Figure 1a. Singly Connected Surface Γ_2 with Boundary C_2. (The entire surface of the region is Γ)

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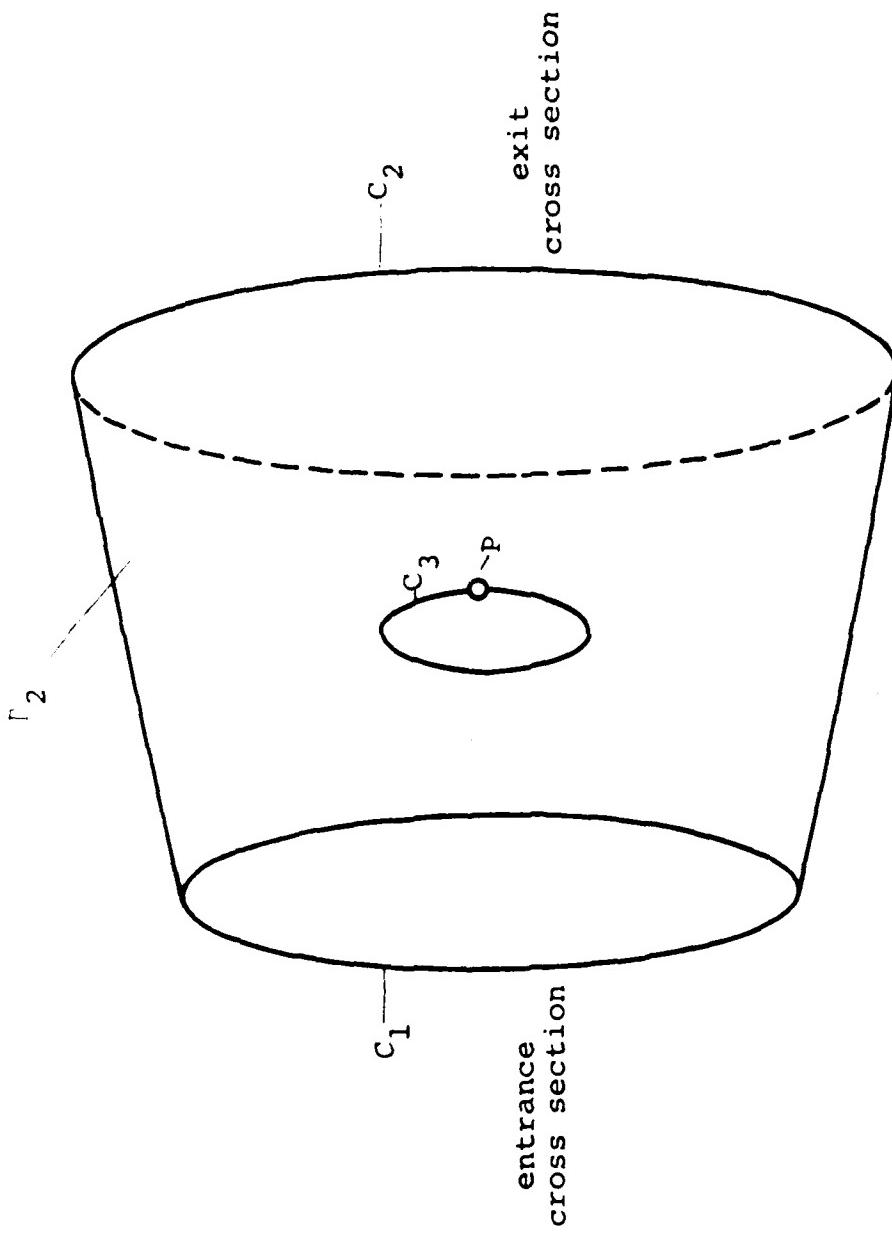
Figure 1b. Surface Γ_2 with Boundary Curves C_1 and C_2 and a Curve C_3 Which Starts and Ends at Point P and Can Be Continuously Contracted into Point P. (The surface Γ is given by Γ_2 plus the entrance and exit cross sections)
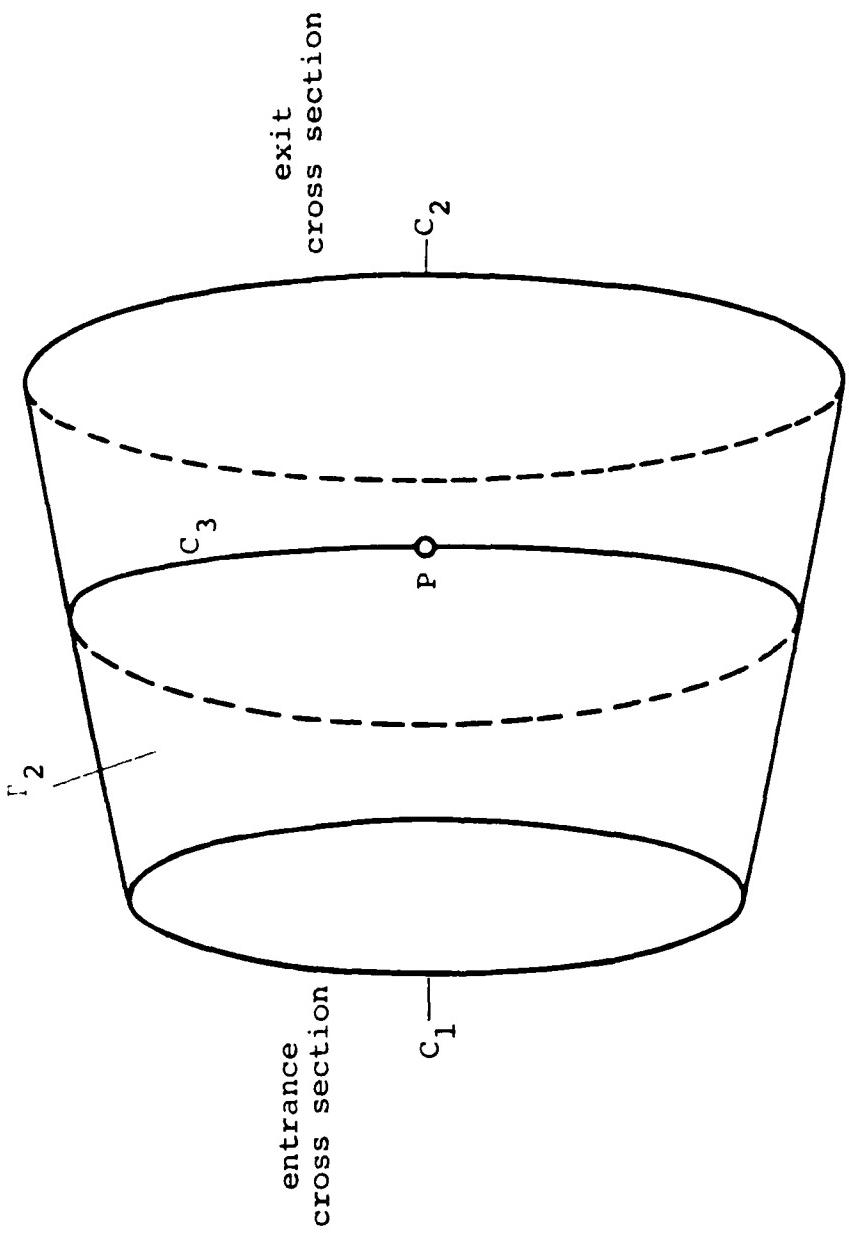
Figure 1c. Surface Γ_2 with Boundary Curves C_1 and C_2 and a Curve C_3 Which Starts at Point P But Cannot Be Continuously Contracted into Point P. (The surface Γ is given by Γ_2 plus the entrance and exit cross sections)

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Γ_2
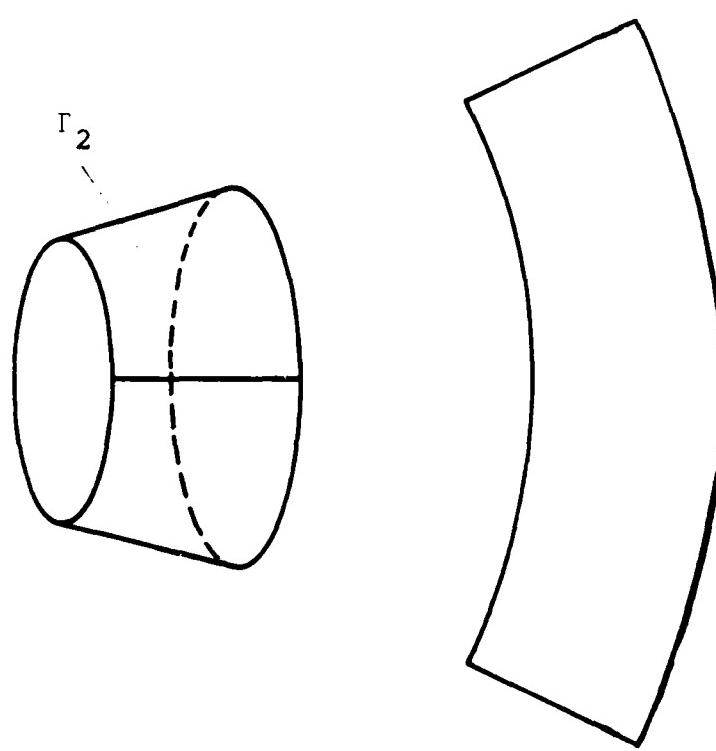
Figure 2. Conical Surface and its Development.

$\psi = C_1$

Γ_2

entrance section

exit section

ϕ prescribed

ϕ prescribed

Γ_2

$\psi = C_2$
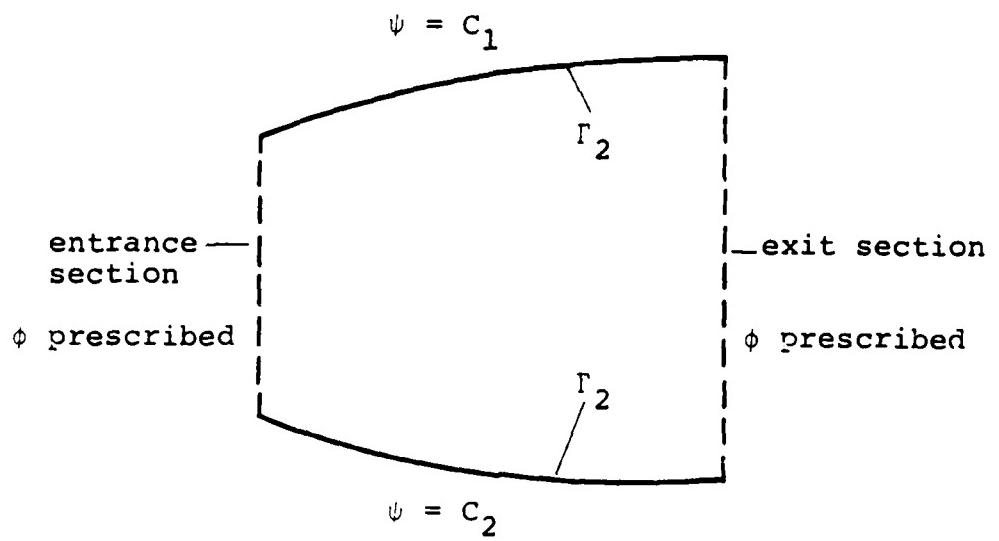
Figure 3.   Two-Dimensional Duct.
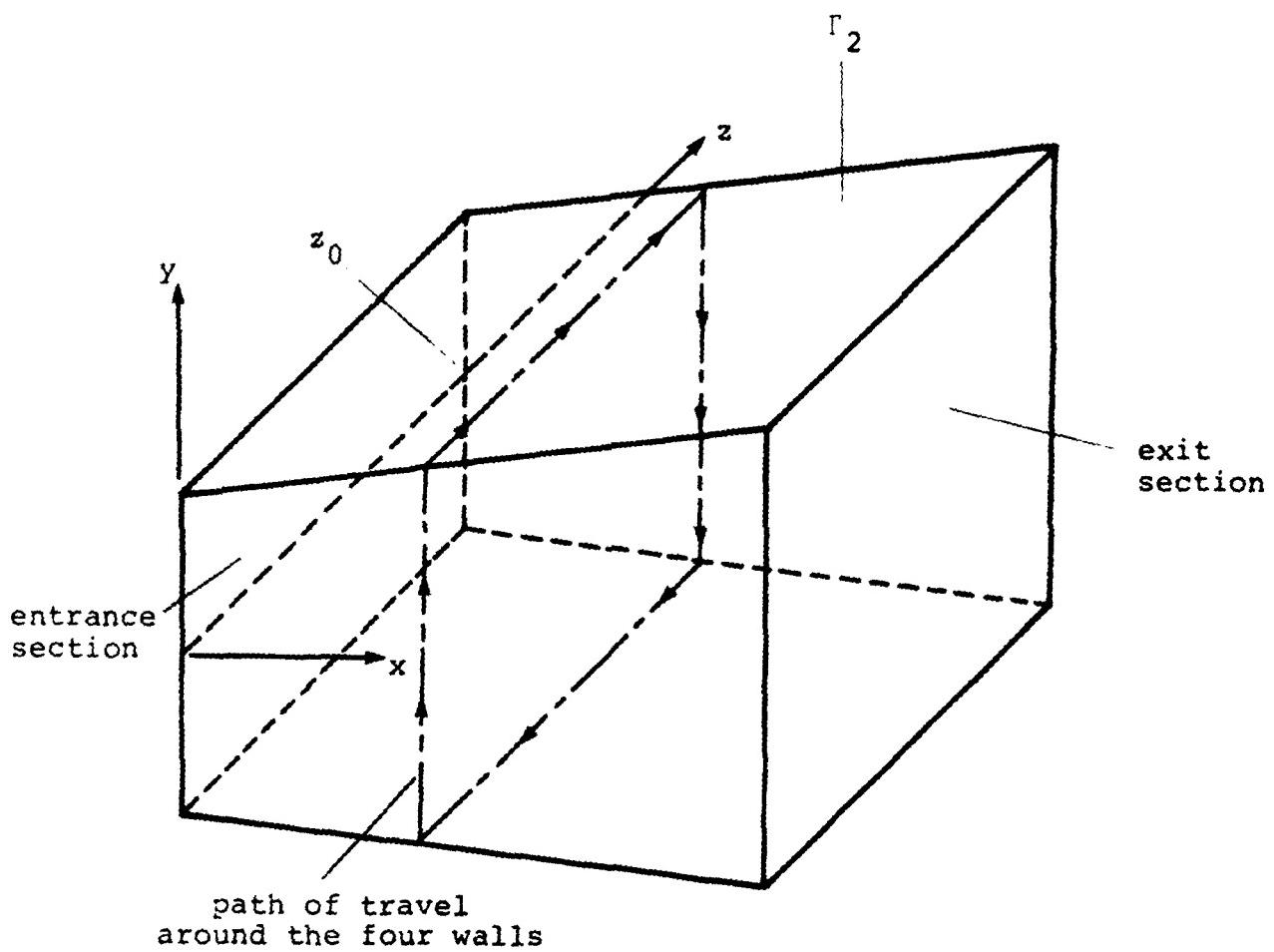
Figure 4. Relation Between the Two-Dimensional and the Three-Dimensional Formulation for the Flow in a Duct.

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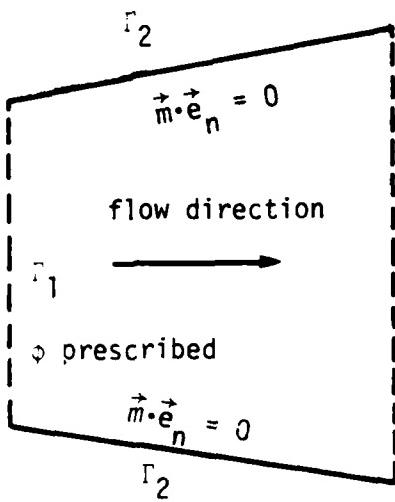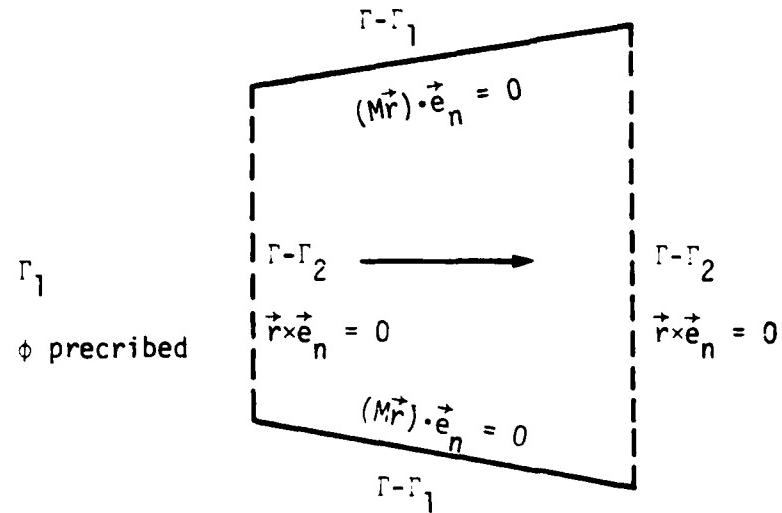
Figure 5. Subsonic Problem: (a) Physical Boundary Value Problem.
(b) Boundary Value Problem for the Residual $\vec{r}$.
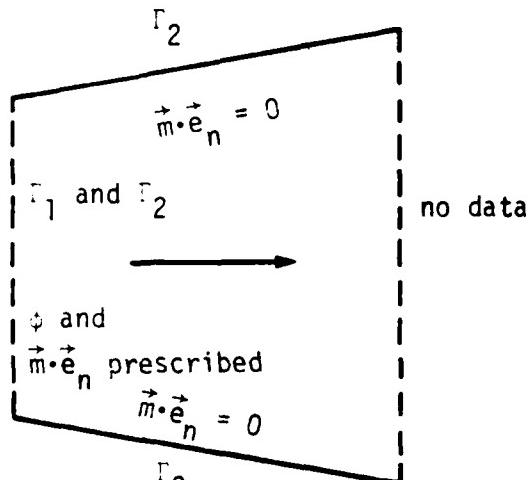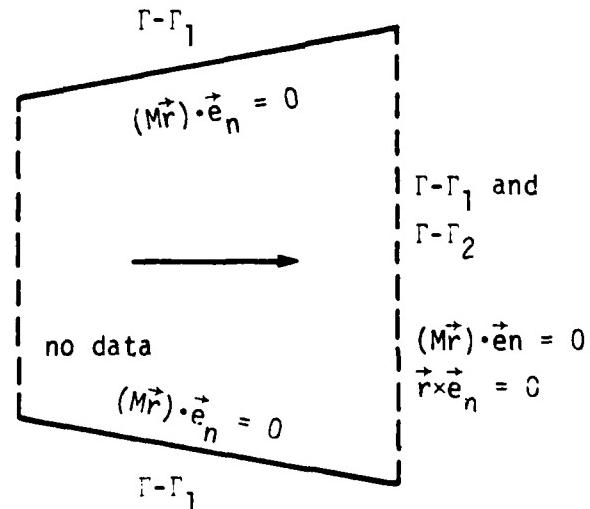
Figure 6. Supersonic Flow: (a) Physical Boundary Value Problem.
(b) Boundary Value Problem for the Residual $\vec{r}$.

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$\vec{m} \cdot \vec{e}_n = 0$

Γ_2

Γ_1 and Γ_2

supersonic

subsonic

Γ_1

shock

ϕ and $\vec{m} \cdot \vec{e}_n$ prescribed

ϕ prescribed

$\vec{m} \cdot \vec{e}_n = 0$

Γ_2

(a)

$\Gamma - \Gamma_1$

$(M\vec{r}) \cdot \vec{e}_n = 0$

$\vec{r} = 0$ carried over from the subsonic region

extra condition for $\vec{r}$ needed at the shock

$\Gamma - \Gamma_2$

$\vec{r} \times \vec{e}_n = 0$

$\Gamma - \Gamma_1$

$(M\vec{r}) \cdot \vec{e}_n = 0$

(b)
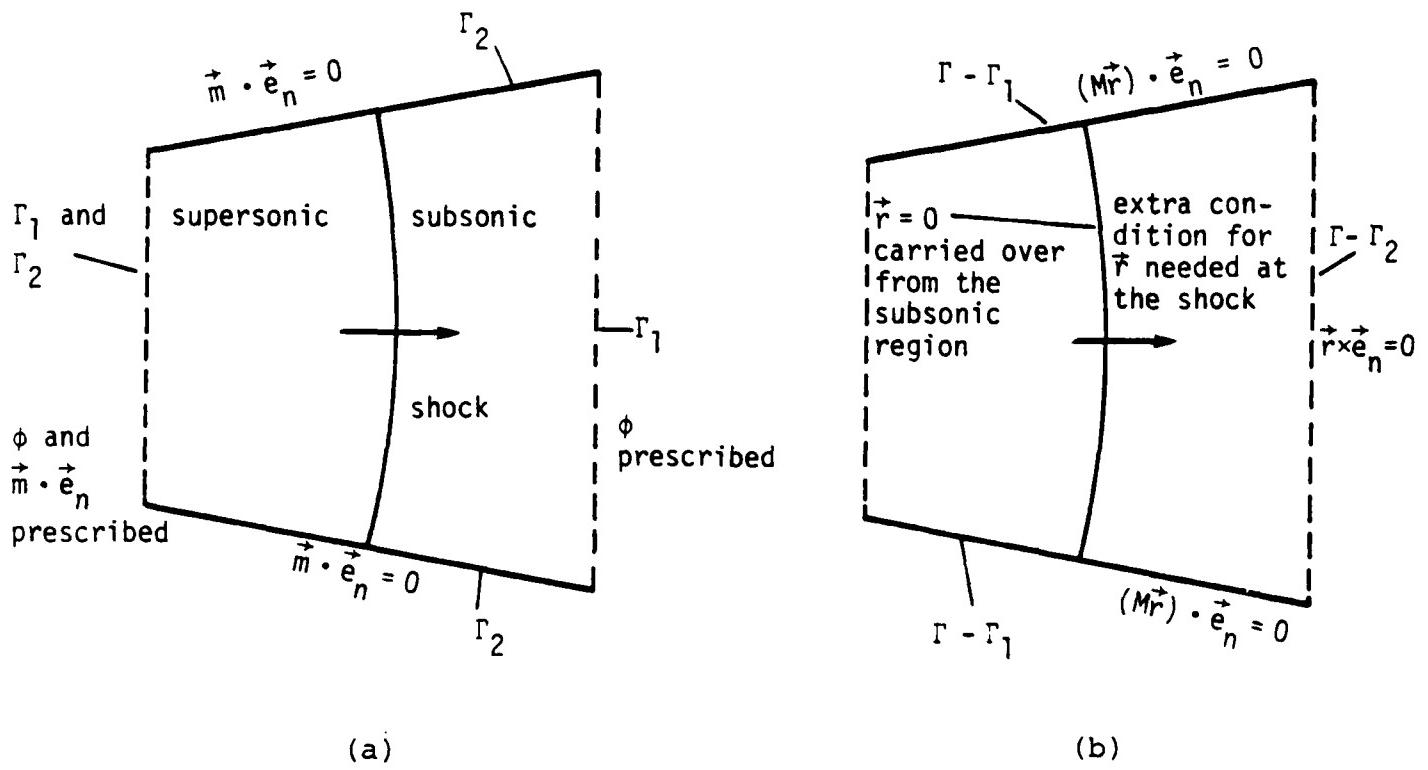
Figure 7. Transition from a Supersonic to a Subsonic Flow Through a Shock. (a) Boundary Value Problem for the Flow Variables. (b) Boundary Value Problem for the Residual $\vec{r}$.
(Along the shock in the subsonic region an additional boundary condition for r is needed, to guarantee that $\vec{r} \equiv 0$ throughout the subsonic region. The shock condition then yields $\vec{r} = 0$ at the shock in the supersonic region)
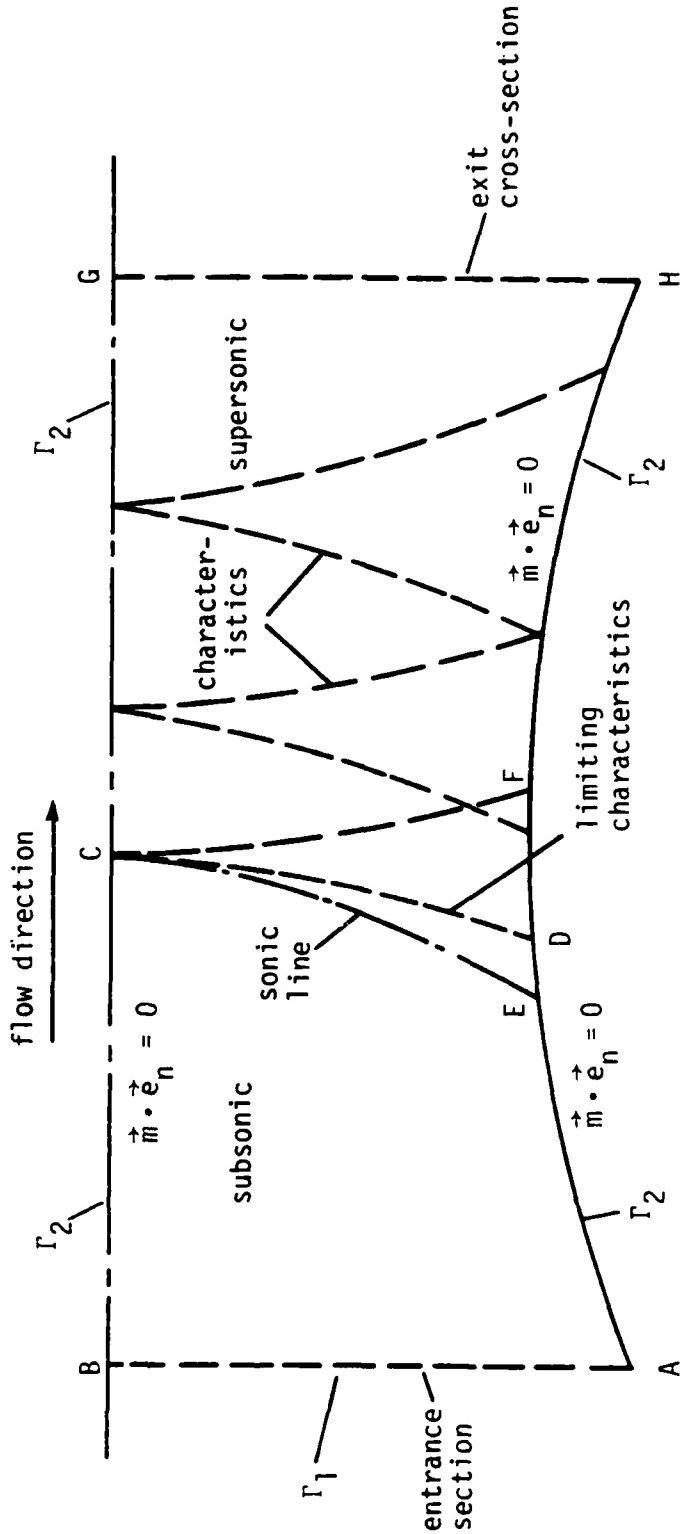
Figure 8a. Transition from a Subsonic to Supersonic Flow in a Laval Nozzle. Boundary Value Problem for the Flow Variables.

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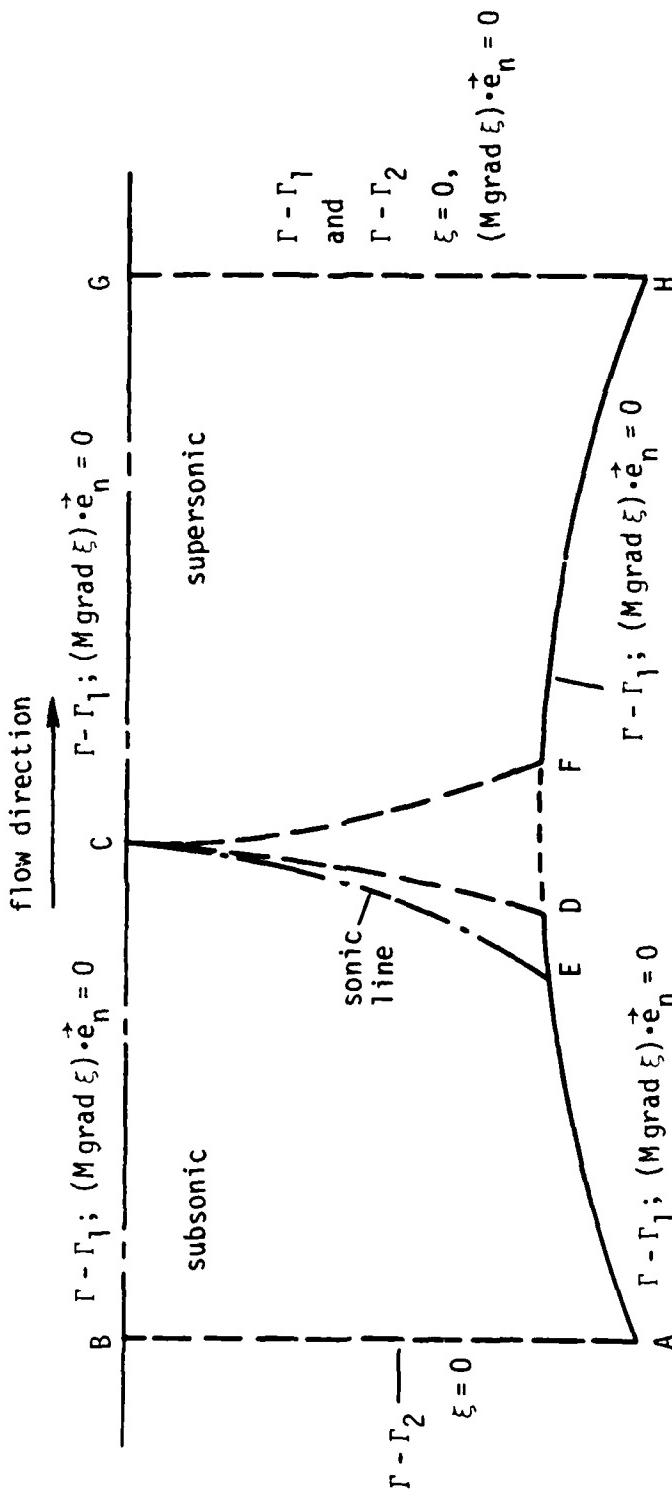
Figure 8b. Transition from a Subsonic to a Supersonic Flow.
Boundary Value Problem for ξ.
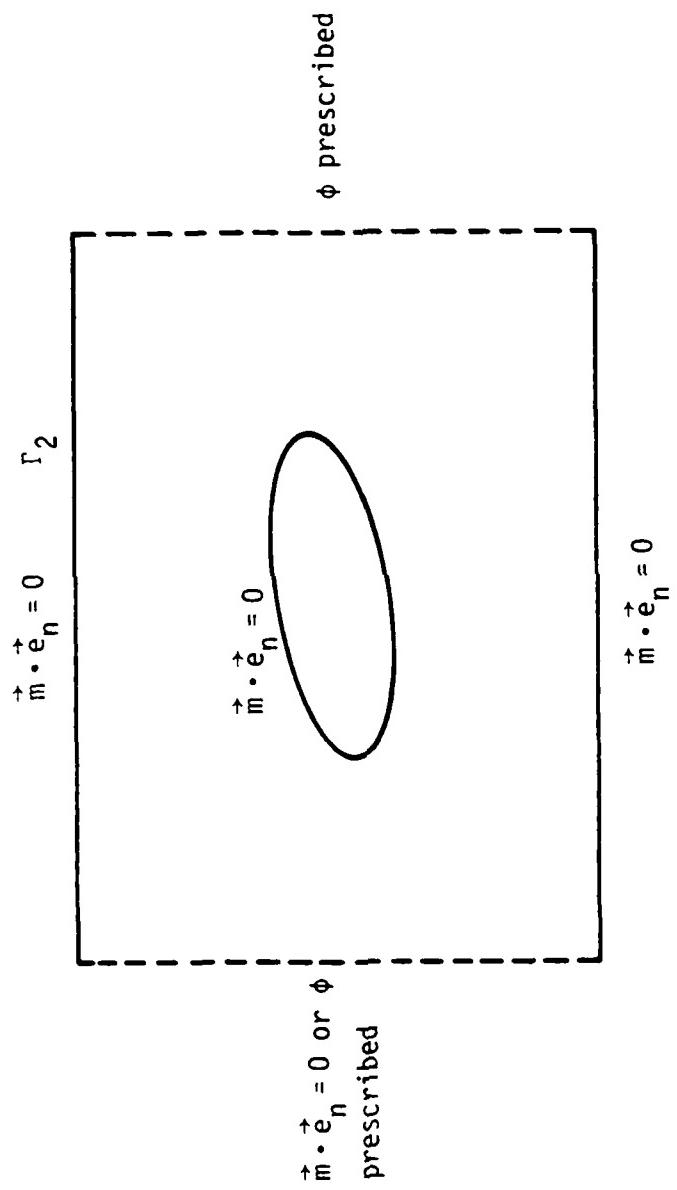
Figure 9. The Boundary Value Problem for the Two-Dimensional Flow in a Wind Tunnel.
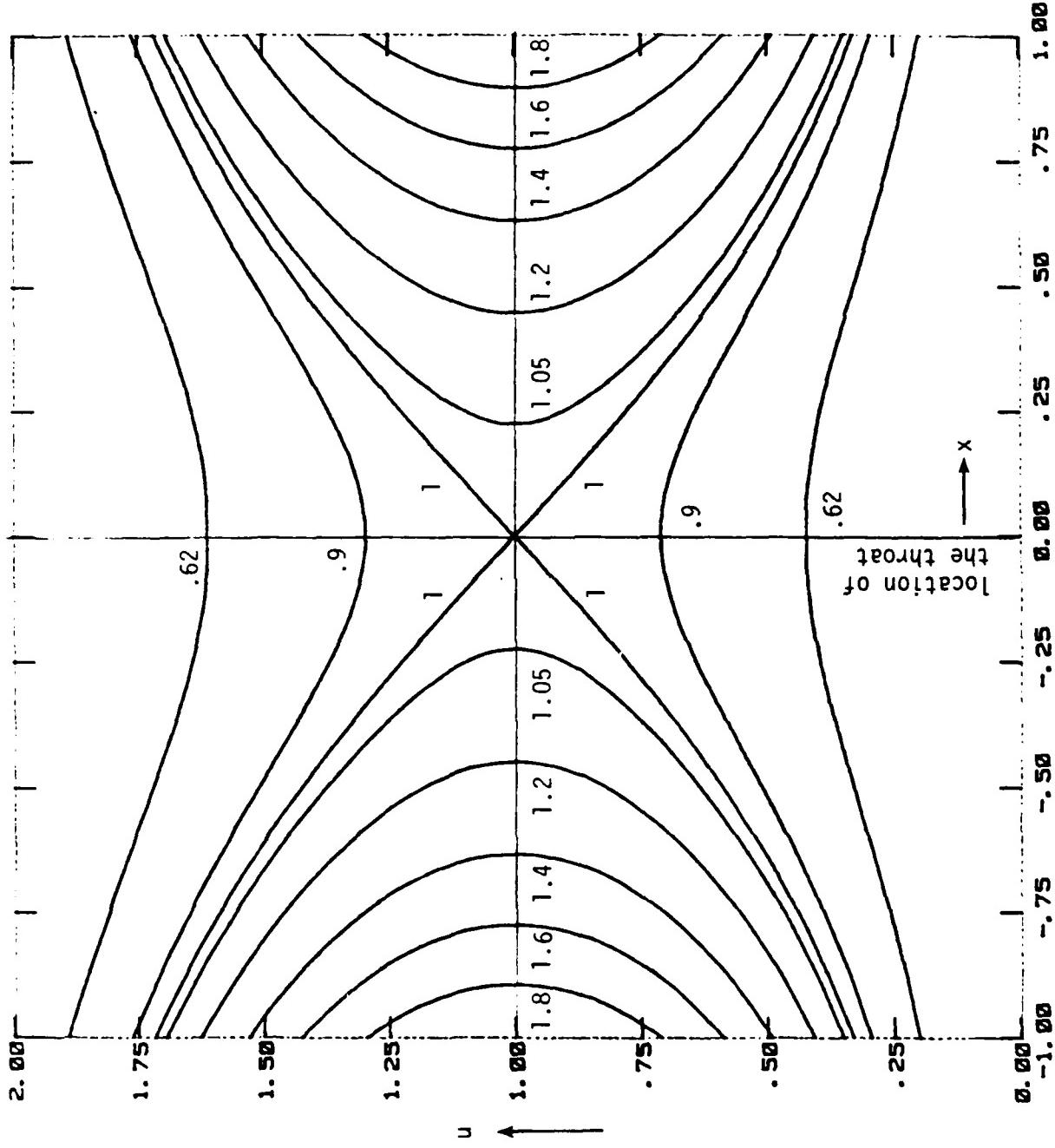
Figure 10. Theoretical Velocity Distribution in a Laval Nozzle without Density Modification.

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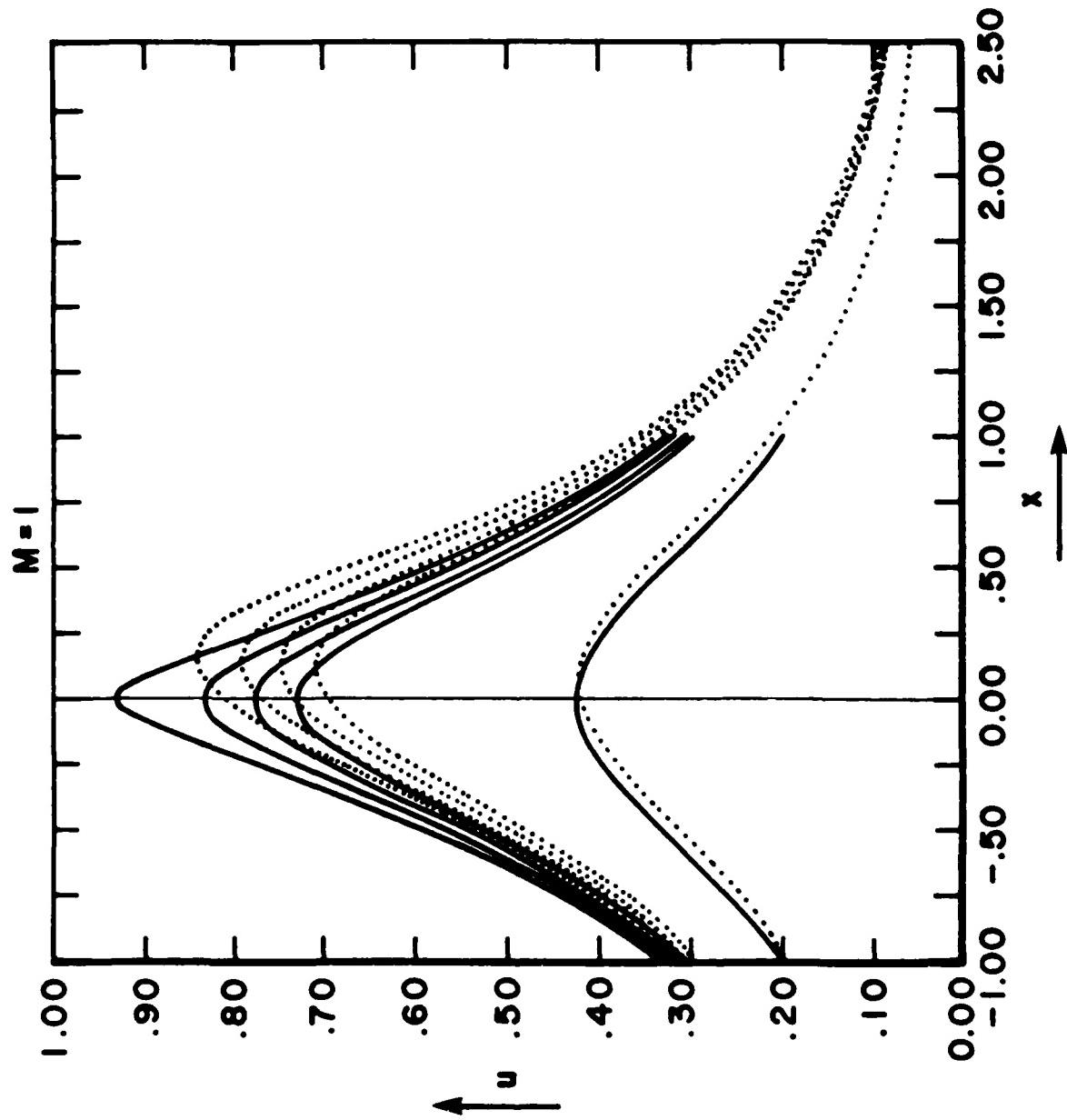
Figure 11. Velocity Distributions for Subsonic Flows in a Laval Nozzle
with (Dashed Curves) and without (Solid Curves) Density
Modification.
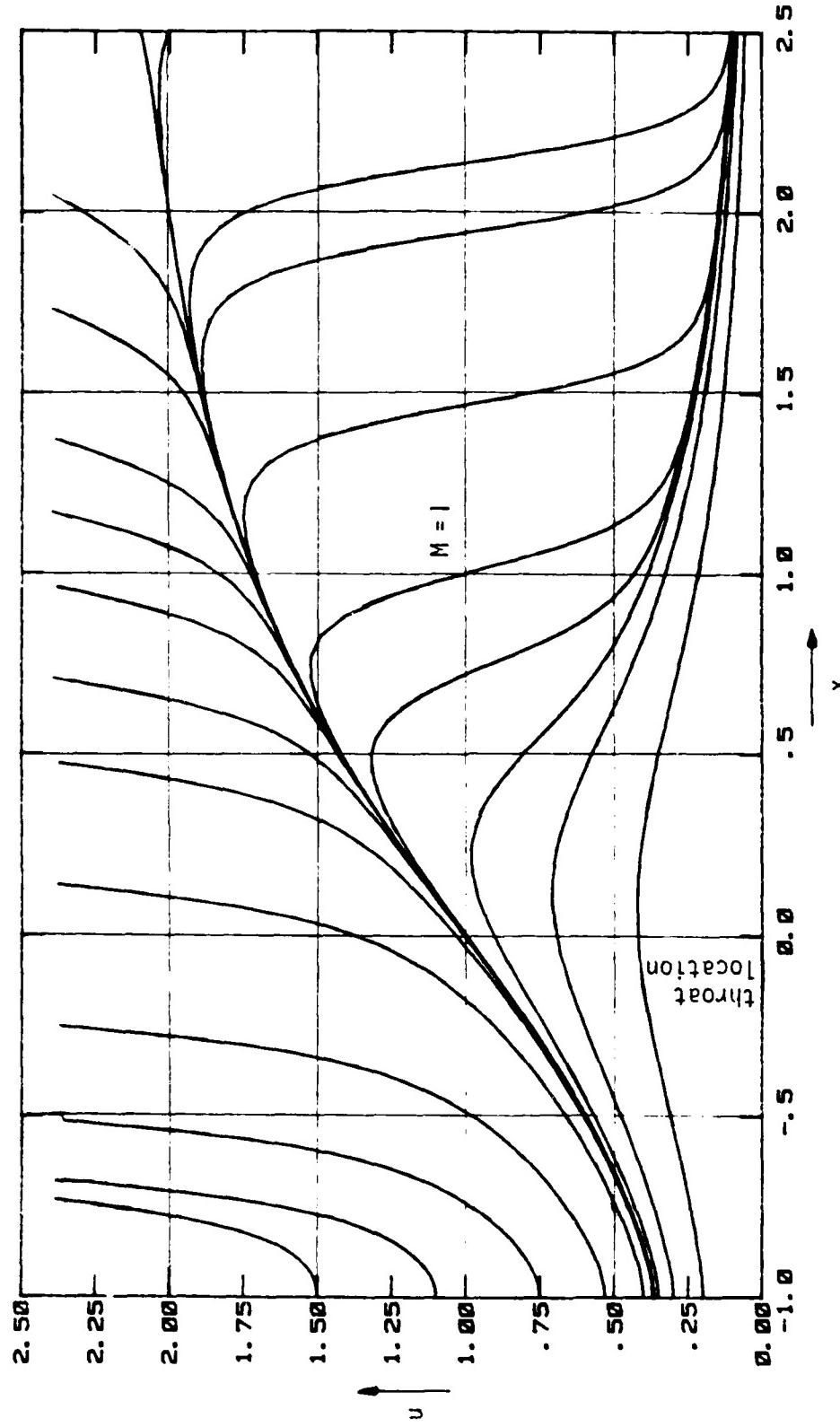
Figure 12. Velocity Distributions for Mixed Flows in a Laval Nozzle with Density Modification.

side wall
(some linear homogeneous boundary
condition for R_1 or R_2)

exit section
(left-going characteristic)

$R_1 + tg(\theta+\alpha)R_2 = 0$

(from Eq. (362))

right-going
characteristics

$R_1 + tg(\theta+\alpha)R_2 = 0$

because of Eq. (366) and
the conditions in the
exit cross section
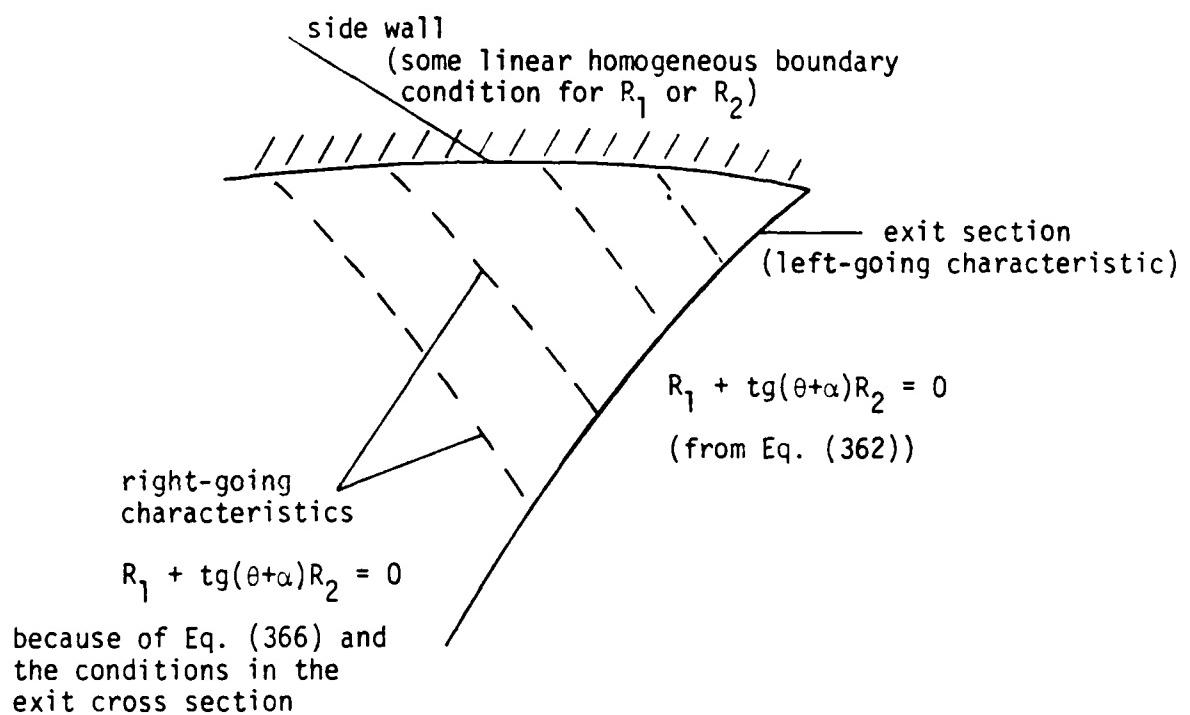
Figure 13.   Conditions for the Residuals R_1 and R_2
if the Exit Cross Section Coincides
with a Characteristic.

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TABLE 1

BRISTEAU METHOD, PHASE ERROR

$M^2 = 1.5$　　$\alpha = 54.74°$　　$P_1/P_2 = .5$

PHASE ERROR (DEGREES) OVER ONE FULL WAVE IN x DIRECTION

θ DEGREES	Δx/Δy	$P_2 = 5°$	10°	15°	30°	45°	60°	90°
-35	.002	.06	.23	.52	2.17	5.27	10.39	30.91
-30	.046	.05	.18	.42	1.76	4.32	8.62	26.30
-25	.091	.03	.12	.28	1.20	3.03	6.23	20.20
-20	.136	.01	.05	.11	.52	1.44	3.30	12.83
-15	.185	-.01	-.04	-.08	-.28	-.39	-.08	4.41
-10	.236	-.03	-.13	-.30	-1.16	-2.43	-3.82	-4.88
-5	.292	-.06	-.24	-.53	-2.10	-4.60	-7.84	-14.89
0	.354	-.09	-.34	-.77	-3.07	-6.86	-12.03	-25.52
5	.423	-.11	-.45	-1.01	-4.05	-9.14	-16.31	-36.64
10	.505	-.14	-.55	-1.24	-5.00	-11.38	-20.57	-48.14
15	.601	-.16	-.65	-1.46	-5.90	-13.53	-24.69	-59.84
20	.721	-.18	-.73	-1.66	-6.72	-15.50	-28.55	-71.47
25	.875	-.20	-.81	-1.83	-7.43	-17.24	-32.01	-82.63
30	1.085	-.22	-.87	-1.96	-8.02	-18.67	-34.91	-92.73
35	1.394	-.23	-.91	-2.07	-8.45	-19.74	-37.12	-100.97

$\dfrac{P_1}{360°}$ and $\dfrac{P_2}{360°}$ are respectively the number of grid points for a full wave in the y and x direction. For the exact solution:

Δx/Δy　Aspect Ratio of the grids

θ = 0　means grid aligned with the flow direction

α =　Mach angle

$P_1/P_2 = .5$,　$M^2 = 1.5$

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TABLE 2

BRISTEAU METHOD, PHASE ERROR

$M^2 = 2.0$ $\qquad \alpha = 45.00°$ $\qquad p_1/p_2 = .5$

PHASE ERROR (DEGREES) OVER ONE FULL WAVE IN x DIRECTION

θ DEGREES	Δx/Δy	$p_2 = 5°$	10°	15°	30°	45°	60°	90°
-40	.044	.06	.22	.50	2.11	5.13	10.12	30.20
-35	.088	.05	.20	.44	1.87	4.57	9.10	27.54
-30	.134	.04	.15	.35	1.48	3.68	7.43	23.26
-25	.182	.02	.10	.22	.96	2.46	5.19	17.56
-20	.233	.01	.03	.06	.31	.97	2.43	10.65
-15	.289	-.01	-.06	-.12	-.44	-.76	-.76	2.73
-10	.350	-.04	-.15	-.33	-1.27	-2.68	-4.29	-6.03
-5	.420	-.06	-.24	-.55	-2.15	-4.73	-8.08	-15.49
0	.500	-.09	-.34	-.77	-3.07	-6.86	-12.03	-25.52
5	.596	-.11	-.44	-1.00	-3.99	-9.01	-16.06	-35.99
10	.714	-.13	-.54	-1.21	-4.89	-11.12	-20.07	-46.76
15	.866	-.16	-.63	-1.42	-5.74	-13.14	-23.94	-57.64
20	1.072	-.18	-.71	-1.60	-6.51	-14.99	-27.55	-68.36
25	1.374	-.20	-.78	-1.77	-7.18	-16.61	-30.77	-78.52
30	1.866	-.21	-.84	-1.90	-7.73	-17.96	-33.46	-87.59
35	2.836	-.22	-.88	-1.99	-8.13	-18.96	-35.50	-94.88
40	5.715	-.23	-.91	-2.05	-8.38	-19.58	-36.78	-99.66

For explanation of the notations see Table 1.

$p_1/p_2 = .5,$ $\qquad M^2 = 2$

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TABLE 3

BRISTEAU METHOD, PHASE ERROR

$M^2 = 2.5$ $\alpha = 39.23°$ $p_1/p_2 = .5$

PHASE ERROR (DEGREES) OVER ONE FULL WAVE IN x DIRECTION

θ DEGREES	Δx/Δy	$p_2 = 5°$	10°	15°	30°	45°	60°	90°
-50	.007	.06	.23	.53	2.21	5.37	10.58	31.41
-45	.051	.06	.24	.55	2.30	5.56	10.94	32.34
-40	.095	.06	.23	.53	2.21	5.37	10.58	31.41
-35	.141	.05	.21	.47	1.97	4.81	9.54	28.68
-30	.190	.04	.16	.37	1.58	3.89	7.84	24.29
-25	.241	.03	.11	.24	1.04	2.65	5.54	18.46
-20	.298	.01	.03	.08	.38	1.13	2.73	11.39
-15	.360	-.01	-.05	-.11	-.38	-.64	-.53	3.30
-10	.431	-.04	-.14	-.32	-1.23	-2.59	-4.13	-5.64
-5	.514	-.06	-.24	-.54	-2.13	-4.68	-8.00	-15.29
0	.612	-.09	-.34	-.77	-3.07	-6.86	-12.03	-25.52
5	.735	-.11	-.44	-1.00	-4.01	-9.05	-16.15	-36.21
10	.893	-.14	-.54	-1.22	-4.93	-11.21	-20.24	-47.23
15	1.111	-.16	-.64	-1.43	-5.79	-13.27	-24.19	-58.38
20	1.433	-.18	-.72	-1.62	-6.58	-15.16	-27.89	-69.41
25	1.971	-.20	-.79	-1.79	-7.27	-16.82	-31.19	-79.90
30	3.076	-.21	-.85	-1.92	-7.83	-18.20	-33.95	-89.31
35	6.758	-.22	-.89	-2.02	-8.24	-19.22	-36.05	-96.90
40	-37.278	-.23	-.92	-2.08	-8.50	-19.86	-37.36	-101.90
45	-4.950	-.23	-.93	-2.10	-8.58	-20.07	-37.81	-103.65
50	-2.629	-.23	-.92	-2.08	-8.50	-19.86	-37.36	-101.90

For explanation of the notations see Table 1.

$p_1/p_2 = .5$, $M^2 = 2.5$

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TABLE 4

BRISTEAU METHOD, PHASE ERROR

$M^2 = 1.5$ $\alpha = 54.74°$ $p_1/p_2 = 1.0$

PHASE ERROR (DEGREES) OVER ONE FULL WAVE IN x DIRECTION

θ DEGREES	$\Delta x/\Delta y$	$p_2 = 5°$	10°	15°	30°	45°	60°	90°
-35	.005	.23	.91	2.04	8.02	17.61	30.42	64.83
-30	.092	.21	.84	1.88	7.40	16.26	28.10	59.87
-25	.181	.19	.74	1.66	6.55	14.43	24.97	53.26
-20	.273	.16	.62	1.40	5.51	12.16	21.10	45.20
-15	.369	.12	.48	1.09	4.30	9.52	16.58	35.83
-10	.472	.08	.33	.74	2.95	6.56	11.51	25.22
-5	.584	.04	.17	.38	1.51	3.37	5.95	13.33
0	.707	.00	.00	.00	.00	.00	.00	.00
5	.847	-.04	-.17	-.38	-1.53	-3.47	-6.27	-15.10
10	1.009	-.08	-.33	-.75	-3.03	-6.95	-12.74	-32.59
15	1.203	-.12	-.49	-1.10	-4.46	-10.34	-19.31	-53.75
20	1.442	-.16	-.63	-1.41	-5.78	-13.55	-25.77	-81.91
25	1.751	-.19	-.75	-1.69	-6.93	-16.44	-31.88	-139.22
30	2.171	-.21	-.84	-1.91	-7.89	-18.88	-37.32	----
35	2.787	-.23	-.92	-2.07	-8.60	-20.74	-41.66	----

For explanation of the notations see Table 1.
Dashes indicate instability.

$p_1/p_2 = 1$, $M^2 = 1.5$

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TABLE 5

BRISTEAU METHOD, PHASE ERROR

$M^2 = 2.0$ $\alpha = 45.00°$ $p_1/p_2 = 1.0$

PHASE ERROR (DEGREES) OVER ONE FULL WAVE IN x DIRECTION

θ DEGREES	Δx/Δy	$p_2 = 5°$	10°	15°	30°	45°	60°	90°
-40	.087	.22	.90	2.01	7.93	17.41	30.06	64.07
-35	.176	.21	.86	1.92	7.57	16.63	28.73	61.20
-30	.268	.20	.79	1.77	6.98	15.35	26.55	56.58
-25	.364	.17	.70	1.57	6.18	13.62	23.59	50.39
-20	.466	.15	.59	1.32	5.20	11.48	19.94	42.80
-15	.577	.11	.46	1.02	4.06	8.98	15.67	33.94
-10	.700	.08	.31	.70	2.79	6.20	10.87	23.88
-5	.839	.04	.16	.36	1.42	3.18	5.62	12.60
0	1.000	-.00	.00	.00	.00	-.00	0.00	-.00
5	1.192	-.04	-.16	-.36	-1.44	-3.27	-5.90	-14.17
10	1.428	-.08	-.31	-.71	-2.85	-6.54	-11.97	-30.35
15	1.732	-.11	-.46	-1.03	-4.20	-9.72	-18.08	-49.40
20	2.145	-.15	-.59	-1.33	-5.44	-12.71	-24.05	-73.22
25	2.747	-.18	-.70	-1.59	-6.52	-15.39	-29.64	-108.57
30	3.732	-.20	-.79	-1.80	-7.41	-17.65	-34.57	----
35	5.671	-.21	-.86	-1.95	-8.08	-19.37	-38.46	----
40	11.430	-.23	-.90	-2.05	-8.49	-20.45	-40.98	----

For explanation of the notations see Table 1.
Dashes indicate instability.

$p_1/p_2 = 1$, $M^2 = 2$

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TABLE 6

BRISTEAU METHOD, PHASE ERROR

$M^2 = 2.5$ $\alpha = 39.23°$ $p_1/p_2 = 1.0$

PHASE ERROR (DEGREES) OVER ONE FULL WAVE IN x DIRECTION

θ DEGREES	Δx/Δy	$p_2 = 5°$	10°	15°	30°	45°	60°	90°
-50	.013	.23	.92	2.06	8.09	17.76	30.67	65.37
-45	.101	.23	.93	2.09	8.21	18.03	31.13	66.37
-40	.190	.23	.92	2.06	8.09	17.76	30.67	65.37
-35	.282	.22	.87	1.96	7.72	16.96	29.30	62.43
-30	.379	.20	.81	1.81	7.12	15.66	27.08	57.70
-25	.483	.18	.71	1.60	6.31	13.89	24.06	51.36
-20	.595	.15	.60	1.34	5.31	11.71	20.33	43.61
-15	.720	.12	.47	1.05	4.14	9.17	15.98	34.58
-10	.862	.08	.32	.72	2.84	6.32	11.09	24.34
-5	1.027	.04	.16	.36	1.45	3.24	5.73	12.85
0	1.225	.00	.00	.00	.00	0.00	-.00	-.00
5	1.470	-.04	-.16	-.37	-1.47	-3.33	-6.02	-14.49
10	1.787	-.08	-.32	-.72	-2.91	-6.68	-12.23	-31.11
15	2.222	-.12	-.47	-1.05	-4.29	-9.93	-18.49	-50.85
20	2.867	-.15	-.60	-1.36	-5.55	-12.99	-24.63	-76.02
25	3.943	-.18	-.72	-1.62	-6.66	-15.75	-30.40	-116.15
30	6.153	-.20	-.81	-1.84	-7.58	-18.07	-35.49	----
35	13.516	-.22	-.88	-1.99	-8.26	-19.83	-39.53	----
40	-74.556	-.23	-.92	-2.09	-8.68	-20.94	-42.15	----
45	-9.899	-.23	-.94	-2.12	-8.82	-21.32	-43.06	----
50	-5.258	-.23	-.92	-2.09	-8.68	-20.94	-42.15	----

For explanation of the notations see Table 1.
Dashes indicate instability.

$p_1/p_2 = 1$, $M^2 = 2.5$

TABLE

REISTFAD METHOD, PHASE ...

$$M^2 = 1.5 \qquad \alpha = 54.74^\circ \qquad P_1/P_2 = 2.0$$

PHASE ERROR (DEGREES) OVER ONE FULL WAVE IN x DIRECTION

θ DEGREES	$\Delta x/\Delta y$	$P_2 = 5^\circ$	10°	15°	30°	45°	60°	90°
-35	.009	.91	3.58	7.90	28.43	55.24	83.24	134.53
-30	.184	.86	3.41	7.52	27.13	52.85	79.81	129.06
-25	.362	.80	3.17	7.00	25.36	49.59	75.10	121.67
-20	.546	.73	2.88	6.36	23.15	45.50	69.17	112.36
-15	.738	.64	2.54	5.62	20.56	40.62	62.03	100.80
-10	.944	.55	2.17	4.80	17.63	35.00	53.57	86.02
-5	1.167	.45	1.77	3.92	14.42	28.65	43.58	64.93
0	1.414	.34	1.35	2.99	10.99	21.54	31.59	.19
5	1.694	.24	.94	2.06	7.40	13.64	16.58	---
10	2.019	.13	.53	1.15	3.73	4.87	-4.14	---
15	2.406	.04	.14	.28	.08	-4.85	-45.68	---
20	2.885	-.05	-.21	-.52	3.43	-15.66	---	---
25	3.501	-.12	-.51	-1.21	6.66	-27.72	---	---
30	4.341	-.18	-.76	-1.79	9.44	-41.12	---	---
35	5.575	-.23	-.94	-2.21	-11.60	-55.73	---	---

For explanation of the notations see Table 1.
Dashes indicate instability.

$$P_1/P_2 = 2, \qquad M^2 = 1.5$$

TABLE 8

BRISTEAU METHOD, PHASE ERROR

$M^2 = 2.0$ $\alpha = 45.00°$ $p_1/p_2 = 2.0$

PHASE ERROR (DEGREES) OVER ONE FULL WAVE IN x DIRECTION

θ DEGREES	$\Delta x/\Delta y$	$p_2 = 5°$	10°	15°	30°	45°	60°	90°
-40	.175	.90	3.56	7.84	28.23	54.88	82.72	133.69
-35	.353	.88	3.46	7.62	27.48	53.50	80.74	130.53
-30	.536	.83	3.29	7.26	26.25	51.24	77.48	125.41
-25	.728	.78	3.07	6.78	24.58	48.14	73.01	118.40
-20	.933	.71	2.80	6.17	22.49	44.26	67.37	109.49
-15	1.155	.63	2.48	5.48	20.04	39.63	60.54	98.32
-10	1.400	.54	2.12	4.70	17.26	34.29	52.47	83.95
-5	1.678	.44	1.74	3.86	14.23	28.26	42.95	63.35
0	2.000	.34	1.35	2.99	10.99	21.54	31.59	.19
5	2.384	.24	.96	2.12	7.61	14.12	17.56	---
10	2.856	.15	.58	1.25	4.17	5.96	-1.18	---
15	3.464	.06	.21	.44	.76	-2.95	-33.11	---
20	4.289	-.03	-.12	-.31	-2.50	-12.63	---	---
25	5.495	-.10	-.40	-.97	-5.49	-23.03	---	---
30	7.464	-.15	-.63	-1.50	-8.04	-33.89	---	---
35	11.343	-.20	-.81	-1.90	-10.02	-44.48	---	---
40	22.860	-.22	-.91	-2.15	-11.27	-53.04	---	---

For explanation of the notations see Table 1.
Dashes indicate instability.

$p_1/p_2 = 2$, $M^2 = 2$

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TABLE 9

BRISTEAU METHOD, PHASE ERROR

$M^2 = 2.5$ $\alpha = 39.23°$ $p_1/p_2 = 2.0$

PHASE ERROR (DEGREES) OVER ONE FULL WAVE IN x DIRECTION

θ DEGREES	Δx/Δy	p_2 = 5°	10°	15°	30°	45°	60°	90°
-50	.027	.91	3.60	7.94	28.57	55.49	83.61	135.12
-45	.202	.92	3.64	8.01	28.83	55.97	84.29	136.22
-40	.380	.91	3.60	7.94	28.57	55.49	83.61	135.12
-35	.565	.89	3.50	7.71	27.80	54.09	81.59	131.89
-30	.758	.84	3.33	7.35	26.55	51.79	78.27	126.65
-25	.965	.79	3.11	6.85	24.84	48.64	73.72	119.52
-20	1.191	.71	2.83	6.24	22.72	44.68	67.98	110.47
-15	1.441	.63	2.50	5.53	20.21	39.97	61.05	99.17
-10	1.724	.54	2.14	4.73	17.39	34.53	52.84	84.66
-5	2.054	.44	1.75	3.88	14.29	28.39	43.16	63.90
0	2.449	.34	1.35	2.99	10.99	21.54	31.59	.19
5	2.939	.24	.95	2.10	7.53	13.95	17.23	----
10	3.574	.14	.56	1.22	4.02	5.60	-2.16	----
15	4.444	.05	.19	.38	.53	-3.59	-36.77	----
20	5.733	-.03	-.15	-.38	-2.82	-13.64	----	----
25	7.886	-.11	-.44	-1.05	-5.88	-24.56	----	----
30	12.305	-.16	-.68	-1.60	-8.52	-36.19	----	----
35	27.031	-.21	-.85	-2.01	-10.55	-47.89	----	----
40	-149.112	-.23	-.96	-2.26	-11.84	-57.79	----	----
45	-19.798	-.24	-1.00	-2.34	-12.29	-62.02	----	----
50	-10.516	-.23	-.96	-2.26	-11.84	-57.79	----	----

For explanation of the notations see Table 1.
Dashes indicate instability.

$p_1/p_2 = 2$, $M^2 = 2.5$

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TABLE 10

BRISTEAU METHOD, PHASE ERROR

$M^2 = 1.5$ $\alpha = 54.74°$ $\Delta x/\Delta y = .707$

PHASE ERROR IN DEGREES AFTER ONE FULL WAVE IN x DIRECTION

θ DEGREES	P_1/P_2	$P_2 = 5°$	$10°$	$15°$	$30°$	$45°$	$60°$	$90°$
-35	153.219	283.22	319.52	332.50	345.92	360.63	360.66	360.76
-30	7.674	12.11	41.24	75.20	157.20	205.92	236.04	270.15
-25	3.905	3.09	11.78	24.69	73.23	118.05	153.08	199.98
-20	2.591	1.25	4.88	10.64	36.57	67.31	96.41	142.68
-15	1.915	.59	2.32	5.13	18.90	37.70	58.11	96.03
-10	1.498	.28	1.10	2.45	9.36	19.62	31.88	57.89
-5	1.212	.11	.42	.94	3.68	7.95	13.42	26.38
0	1.000	.00	.00	.00	.00	.00	.00	.00
5	.835	-.07	-.28	-.63	-2.53	-5.68	-10.07	-22.36
10	.701	-.12	-.48	-1.08	-4.35	-9.89	-17.81	-41.46
15	.588	-.16	-.63	-1.41	-5.71	-13.08	-23.87	-57.79
20	.490	-.18	-.74	-1.66	-6.73	-15.53	-28.61	-71.60
25	.404	-.20	-.82	-1.84	-7.50	-17.39	-32.28	-82.97
30	.326	-.22	-.88	-1.98	-8.06	-18.75	-35.01	-91.89
35	.254	-.23	-.92	-2.07	-8.44	-19.69	-36.89	-98.26

For meaning of P_1 and P_2 see Table 1.

Aspect Ratio $\Delta x/\Delta y$ = const, chosen so that at $\theta = 0$ the mesh Courant Number is 1.

$M^2 = 1.5$

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TABLE 11

BRISTEAU METHOD, PHASE ERROR

$M^2 = 2.0$ $\alpha = 45.00°$ $\Delta x/\Delta y = 1.000$

PHASE ERROR IN DEGREES AFTER ONE FULL WAVE IN x DIRECTION

θ DEGREES	P_1/P_2	$P_2 = 5°$	10°	15°	30°	45°	60°	90°
-40	11.430	25.87	76.54	123.17	208.99	250.71	274.56	300.51
-35	5.671	6.87	24.97	48.99	120.25	171.14	205.50	247.15
-30	3.732	2.91	11.15	23.46	70.61	115.19	150.67	199.23
-25	2.747	1.48	5.80	12.59	42.39	76.43	107.67	156.18
-20	2.145	.82	3.23	7.11	25.57	49.52	74.21	117.52
-15	1.732	.46	1.81	4.02	15.03	30.61	48.23	82.87
-10	1.428	.24	.94	2.11	8.10	17.10	28.04	51.91
-5	1.192	.10	.38	.86	3.35	7.27	12.31	24.36
0	1.000	-.00	.00	.00	.00	-.00	0.00	-.00
5	.839	-.07	-.27	-.61	-2.43	-5.45	-9.65	-21.36
10	.700	-.12	-.47	-1.05	-4.22	-9.58	-17.23	-39.8
15	.577	-.15	-.61	-1.38	-5.56	-12.72	-23.16	-55.69
20	.466	-.18	-.72	-1.62	-6.56	-15.10	-27.76	-68.90
25	.364	-.20	-.79	-1.79	-7.30	-16.88	-31.26	-79.60
30	.268	-.21	-.85	-1.92	-7.83	-18.18	-33.83	-87.85
35	.176	-.22	-.89	-2.01	-8.18	-19.05	-35.59	-93.70
40	.087	-.23	-.91	-2.05	-8.39	-19.56	-36.62	-97.20

For explanation of the notations see Table 10.

$M^2 = 2$

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TABLE 12

BRISTEAU METHOD, PHASE ERROR

$M^2 = 2.5$ $\alpha = 39.23°$ $\Delta x/\Delta y = 1.225$

PHASE ERROR IN DEGREES AFTER ONE FULL WAVE IN x DIRECTION

θ DEGREES	p_1/p_2	$p_2 = 5°$	10°	15°	30°	45°	60°	90°
-50	91.303	240.57	294.97	315.51	337.29	344.83	348.67	352.59
-45	12.124	29.14	83.83	132.12	217.63	258.17	281.19	306.33
-40	6.440	9.07	32.10	61.02	139.18	190.39	223.61	263.11
-35	4.337	4.12	15.53	31.95	89.48	138.37	174.79	222.46
-30	3.230	2.21	8.54	18.24	57.96	98.93	133.67	184.10
-25	2.537	1.28	5.01	10.91	37.54	69.18	99.27	147.86
-20	2.057	.76	3.00	6.61	23.93	46.77	70.72	113.71
-15	1.700	.44	1.75	3.89	14.61	29.85	47.20	81.70
-10	1.420	.24	.94	2.10	8.07	17.06	28.00	51.96
-5	1.192	.10	.39	.87	3.39	7.36	12.46	24.67
0	1.000	-.00	-.00	-.00	-.00	0.00	-.00	-.00
5	.833	-.07	-.28	-.62	-2.49	-5.59	-9.89	-21.90
10	.685	-.12	-.48	-1.07	-4.32	-9.82	-17.66	-40.94
15	.551	-.16	-.62	-1.40	-5.68	-13.01	-23.70	-57.13
20	.427	-.18	-.73	-1.65	-6.68	-15.39	-28.32	-70.52
25	.311	-.20	-.81	-1.82	-7.41	-17.16	-31.80	-81.25
30	.199	-.21	-.86	-1.94	-7.93	-18.42	-34.32	-89.44
35	.091	-.22	-.90	-2.03	-8.27	-19.28	-36.05	-95.25
40	-.016	-.23	-.92	-2.08	-8.48	-19.78	-37.07	-98.76
45	-.124	-.23	-.93	-2.09	-8.55	-19.95	-37.42	-100.00
50	-.233	-.23	-.92	-2.08	-8.48	-19.78	-37.09	-98.91

For explanation of the notations see Table 10.

$M^2 = 2.5$

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TABLE 13

φ, ψ METHOD, PHASE ERROR

$M^2 = 1.5$ $\alpha = 54.74°$ $p_1/p_2 = .500$

PHASE ERROR IN DEGREES OVER ONE FULL WAVE IN x DIRECTION

θ DEGREES	$\Delta x/\Delta y$	$p_2 = 5°$	10°	15°	30°	45°	60°
-35	.002	.11	.48	1.09	4.30	9.51	16.55
-30	.046	.12	.49	1.10	4.35	9.63	16.75
-25	.091	.13	.50	1.12	4.42	9.78	17.04
-20	.136	.13	.51	1.14	4.51	9.98	17.39
-15	.185	.13	.52	1.16	4.61	10.21	17.82
-10	.236	.13	.53	1.19	4.72	10.46	18.29
-5	.292	.14	.54	1.22	4.83	10.72	18.77
0	.354	.14	.56	1.25	4.95	10.99	19.24
5	.423	.14	.57	1.28	5.06	11.24	19.67
10	.505	.15	.58	1.31	5.17	11.46	20.03
15	.601	.15	.59	1.33	5.27	11.66	20.32
20	.721	.15	.60	1.36	5.36	11.82	20.53
25	.875	.15	.61	1.38	5.43	11.94	20.69
30	1.085	.16	.62	1.39	5.48	12.04	20.79
35	1.394	.16	.63	1.40	5.52	12.10	20.85

For explanation of the notations see Table 1.

$p_1/p_2 = .5$, $M^2 = 1.5$

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TABLE 14

ϕ, ψ METHOD, PHASE ERROR

$M^2 = 2.0$ $\alpha = 45.00°$ $\rho_1/\rho_2 = .500$

PHASE ERROR IN DEGREES OVER ONE FULL WAVE IN x DIRECTION

θ DEGREES	$\Delta x/\Delta y$	$p_2 = 5°$	$10°$	$15°$	$30°$	$45°$	$60°$
-40	.044	.11	.46	1.03	4.07	9.05	15.88
-35	.088	.11	.46	1.03	4.07	9.05	15.88
-30	.134	.11	.46	1.03	4.07	9.05	15.88
-25	.182	.11	.46	1.03	4.07	9.05	15.88
-20	.233	.11	.46	1.03	4.07	9.05	15.88
-15	.289	.11	.46	1.03	4.07	9.05	15.88
-10	.350	.11	.46	1.03	4.07	9.05	15.88
-5	.420	.11	.46	1.03	4.07	9.05	15.88
0	.500	.11	.46	1.03	4.07	9.05	15.88
5	.596	.11	.46	1.03	4.07	9.05	15.88
10	.714	.11	.46	1.03	4.07	9.05	15.88
15	.866	.11	.46	1.03	4.07	9.05	15.88
20	1.072	.11	.46	1.03	4.07	9.05	15.88
25	1.374	.11	.46	1.03	4.07	9.05	15.88
30	1.866	.11	.46	1.03	4.07	9.05	15.88
35	2.836	.11	.46	1.03	4.07	9.05	15.88
40	5.715	.11	.46	1.03	4.07	9.05	15.88

For explanation of the notations see Table 1.

$\rho_1/\rho_2 = .5$, $M^2 = 2$

TABLE 15

Φ, ψ METHOD, PHASE ERROR

$M^2 = 2.5$ $\alpha = 39.23°$ $p_1/p_2 = .500$

PHASE ERROR IN DEGREES OVER ONE FULL WAVE IN x DIRECTION

θ DEGREES	Δx/Δy	$p_2 = 5°$	10°	15°	30°	45°	60°
-50	.007	.12	.47	1.05	4.15	9.20	16.09
-45	.051	.12	.47	1.05	4.14	9.19	16.08
-40	.095	.12	.47	1.05	4.15	9.20	16.09
-35	.141	.12	.47	1.05	4.16	9.22	16.13
-30	.190	.12	.47	1.05	4.17	9.26	16.20
-25	.241	.12	.47	1.06	4.19	9.31	16.29
-20	.298	.11	.47	1.06	4.22	9.37	16.41
-15	.360	.12	.48	1.07	4.25	9.44	16.54
-10	.431	.12	.48	1.08	4.29	9.53	16.69
-5	.514	.12	.49	1.09	4.32	9.61	16.85
0	.612	.12	.49	1.10	4.36	9.69	17.00
5	.735	.12	.49	1.11	4.40	9.78	17.14
10	.893	.12	.50	1.12	4.43	9.85	17.26
15	1.111	.13	.50	1.13	4.47	9.91	17.36
20	1.433	.13	.51	1.13	4.49	9.97	17.43
25	1.971	.13	.51	1.14	4.52	10.01	17.49
30	3.076	.13	.51	1.15	4.54	10.04	17.53
35	6.758	.13	.51	1.15	4.55	10.07	17.55
40	-37.276	.13	.51	1.15	4.56	10.08	17.56
45	-4.949	.13	.51	1.15	4.56	10.08	17.57
50	-2.629	.13	.51	1.15	4.56	10.08	17.56

For explanation of the notations see Table 1.

$p_1/p_2 = .5$, $M^2 = 2.5$

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TABLE 16

φ, ψ METHOD, PHASE ERROR

$M^2 = 1.5$ $\alpha = 54.74°$ $p_1/p_2 = 1.000$

PHASE ERROR IN DEGREES OVER ONE FULL WAVE IN x DIRECTION

θ DEGREES	Δx/Δy	$p_2 = 5°$	10°	15°	30°	45°	60°
-35	.005	.48	1.92	4.24	15.61	31.00	47.36
-30	.092	.49	1.93	4.26	15.68	31.16	47.61
-25	.181	.49	1.94	4.29	15.78	31.37	47.97
-20	.273	.49	1.95	4.32	15.91	31.63	48.40
-15	.369	.50	1.97	4.36	16.05	31.93	48.89
-10	.472	.50	1.99	4.40	16.20	32.24	49.39
-5	.584	.51	2.01	4.45	16.36	32.54	49.87
0	.707	.51	2.03	4.49	16.51	32.83	50.28
5	.847	.52	2.05	4.53	16.66	33.08	50.59
10	1.009	.52	2.07	4.58	16.79	33.28	50.80
15	1.203	.53	2.09	4.62	16.91	33.43	50.93
20	1.442	.53	2.10	4.65	17.00	33.55	50.99
25	1.751	.54	2.12	4.68	17.08	33.62	51.01
30	2.171	.54	2.13	4.70	17.13	33.67	51.00
35	2.787	.54	2.14	4.72	17.17	33.70	50.98

For explanation of the notations see Table 1.

$p_1/p_2 = 1$, $M^2 = 1.5$

TABLE 17

Φ, ψ METHOD, PHASE ERROR

$M^2 = 2.0$ $\alpha = 45.00°$ $p_1/p_2 = 1.000$

PHASE ERROR IN DEGREES OVER ONE FULL WAVE IN x DIRECTION

θ DEGREES	$\Delta x/\Delta y$	$p_2 = 5°$	10°	15°	30°	45°	60°
-40	.087	.46	1.81	4.00	14.78	29.56	45.53
-35	.176	.46	1.81	4.00	14.78	29.56	45.53
-30	.268	.46	1.81	4.00	14.78	29.56	45.53
-25	.364	.46	1.81	4.00	14.78	29.56	45.53
-20	.466	.46	1.81	4.00	14.78	29.56	45.53
-15	.577	.46	1.81	4.00	14.78	29.56	45.53
-10	.700	.46	1.81	4.00	14.78	29.56	45.53
-5	.839	.46	1.81	4.00	14.78	29.56	45.53
0	1.000	.46	1.81	4.00	14.78	29.56	45.53
5	1.192	.46	1.81	4.00	14.78	29.56	45.53
10	1.428	.46	1.81	4.00	14.78	29.56	45.53
15	1.732	.46	1.81	4.00	14.78	29.56	45.53
20	2.145	.46	1.81	4.00	14.78	29.56	45.53
25	2.747	.46	1.81	4.00	14.78	29.56	45.53
30	3.732	.46	1.81	4.00	14.78	29.56	45.53
35	5.671	.46	1.81	4.00	14.78	29.56	45.53
40	11.430	.46.	1.81	4.00	14.78	29.56	45.53

For explanation of the notations see Table 1.

$p_1/p_2 = 1$, $M^2 = 2$

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TABLE 18

φ, ψ METHOD, PHASE ERROR

$$M^2 = 2.5 \qquad \alpha = 39.23^\circ \qquad p_1/p_2 = 1.000$$

PHASE ERROR IN DEGREES OVER ONE FULL WAVE IN x DIRECTION

θ DEGREES	Δx/Δy	$p_2 = 5^\circ$	10°	15°	30°	45°	60°
-50	.013	.47	1.84	4.08	15.05	30.04	46.13
-45	.101	.47	1.84	4.08	15.05	30.03	46.12
-40	.190	.47	1.84	4.08	15.05	30.04	46.13
-35	.282	.47	1.84	4.08	15.07	30.07	46.19
-30	.379	.47	1.85	4.09	15.09	30.12	46.27
-25	.483	.47	1.85	4.10	15.12	30.19	46.38
-20	.595	.47	1.86	4.11	15.16	30.27	46.52
-15	.720	.47	1.86	4.12	15.21	30.36	46.68
-10	.862	.47	1.87	4.13	15.26	30.46	46.84
-5	1.027	.47	1.87	4.15	15.31	30.56	47.00
0	1.225	.47	1.88	4.16	15.36	30.66	47.13
5	1.470	.48	1.89	4.18	15.41	30.74	47.24
10	1.787	.48	1.89	4.19	15.45	30.81	47.32
15	2.222	.48	1.90	4.20	15.49	30.87	47.37
20	2.867	.48	1.90	4.21	15.52	30.91	47.39
25	3.943	.48	1.91	4.22	15.55	30.93	47.41
30	6.153	.48	1.91	4.23	15.57	30.95	47.41
35	13.516	.48	1.91	4.24	15.58	30.97	47.41
40	-74.553	.48	1.92	4.24	15.59	30.97	47.40
45	-9.899	.48	1.92	4.24	15.59	30.97	47.40
50	-5.258	.48	1.92	4.24	15.59	30.97	47.40

For explanation of the notations see Table 1.

$$p_1/p_2 = 1, \qquad M^2 = 2.5$$

TABLE 19

φ, ψ METHOD, PHASE ERROR

$M^2 = 1.5$ $\alpha = 54.74^\circ$ $p_1/p_2 = 2.000$

PHASE ERROR IN DEGREES OVER ONE FULL WAVE IN x DIRECTION

θ DEGREES	Δx/Δy	$p_2 = 5^\circ$	10°	15°	30°	45°	60°
-35	.009	1.92	7.41	15.76	49.24	82.51	109.37
-30	.184	1.92	7.43	15.80	49.38	82.76	109.73
-25	.362	1.93	7.46	15.86	49.57	83.09	110.21
-20	.546	1.94	7.49	15.94	49.79	83.46	110.73
-15	.738	1.95	7.53	16.02	50.02	83.83	111.23
-10	.944	1.96	7.58	16.11	50.26	84.16	111.62
-5	1.167	1.98	7.62	16.20	50.48	84.42	111.85
0	1.414	1.99	7.67	16.29	50.66	84.57	111.91
5	1.694	2.00	7.71	16.38	50.81	84.64	111.85
10	2.019	2.01	7.76	16.46	50.91	84.63	111.70
15	2.406	2.02	7.80	16.53	50.98	84.57	111.51
20	2.885	2.03	7.83	16.59	51.01	84.48	111.32
25	3.501	2.04	7.86	16.63	51.03	84.39	111.15
30	4.341	2.05	7.88	16.67	51.04	84.31	111.01
35	5.575	2.06	7.90	16.70	51.03	84.25	110.91

For explanation of the notations see Table 1.

$p_1/p_2 = 2$, $M^2 = 1.5$

TABLE 20

ϕ, ψ METHOD, PHASE ERROR

$M^2 = 2.0$ $\alpha = 45.00°$ $p_1/p_2 = 2.000$

PHASE ERROR IN DEGREES OVER ONE FULL WAVE IN x DIRECTION

θ DEGREES	$\Delta x/\Delta y$	$p_2 = 5°$	$10°$	$15°$	$30°$	$45°$	$60°$
-40	.175	1.81	6.99	14.90	47.02	79.59	106.49
-35	.353	1.81	6.99	14.90	47.02	79.59	106.49
-30	.536	1.81	6.99	14.90	47.02	79.59	106.49
-25	.728	1.81	6.99	14.90	47.02	79.59	106.49
-20	.933	1.81	6.99	14.90	47.02	79.59	106.49
-15	1.155	1.81	6.99	14.90	47.02	79.59	106.49
-10	1.400	1.81	6.99	14.90	47.02	79.59	106.49
-5	1.678	1.81	6.99	14.90	47.02	79.59	106.49
0	2.000	1.81	6.99	14.90	47.02	79.59	106.49
5	2.384	1.81	6.99	14.90	47.02	79.59	106.49
10	2.856	1.81	6.99	14.90	47.02	79.59	106.49
15	3.464	1.81	6.99	14.90	47.02	79.59	106.49
20	4.289	1.81	6.99	14.90	47.02	79.59	106.49
25	5.495	1.81	6.99	14.90	47.02	79.59	106.49
30	7.464	1.81	6.99	14.90	47.02	79.59	106.49
35	11.343	1.81	6.99	14.90	47.02	79.59	106.49
40	22.860	1.81	6.99	14.90	47.02	79.59	106.49

For explanation of the notations see Table 1.

$p_1/p_2 = 2$, $M^2 = 2$

TABLE 21

Φ,Ψ METHOD, PHASE ERROR

$M^2 = 2.5$ $\alpha = 39.23°$ $p_1/p_2 = 2.000$

PHASE ERROR IN DEGREES OVER ONE FULL WAVE IN x DIRECTION

θ DEGREES	Δx/Δy	$p_2 = 5°$	10°	15°	30°	45°	60°
-50	.027	1.84	7.13	15.19	47.76	80.56	107.44
-45	.202	1.84	7.12	15.18	47.75	80.54	107.41
-40	.380	1.84	7.13	15.19	47.76	80.56	107.44
-35	.565	1.85	7.13	15.20	47.79	80.61	107.51
-30	.758	1.85	7.14	15.21	47.83	80.69	107.63
-25	.965	1.85	7.15	15.23	47.89	80.79	107.78
-20	1.191	1.85	7.16	15.25	47.96	80.91	107.95
-15	1.441	1.86	7.17	15.28	48.04	81.03	108.11
-10	1.724	1.86	7.18	15.31	48.12	81.14	108.24
-5	2.054	1.86	7.20	15.34	48.19	81.23	108.32
0	2.449	1.87	7.21	15.37	48.25	81.28	108.34
5	2.939	1.87	7.23	15.40	48.30	81.31	108.33
10	3.574	1.88	7.24	15.42	48.34	81.32	108.29
15	4.444	1.88	7.26	15.45	48.37	81.30	108.23
20	5.733	1.88	7.27	15.47	48.38	81.28	108.17
25	7.886	1.89	7.28	15.48	48.39	81.25	108.11
30	12.305	1.89	7.28	15.49	48.39	81.22	108.07
35	27.031	1.89	7.29	15.50	48.40	81.20	108.03
40	-149.106	1.89	7.29	15.51	48.40	81.19	108.01
45	-19.798	1.89	7.29	15.51	48.40	81.19	108.00
50	-10.516	1.89	7.29	15.51	48.40	81.19	108.01

For explanation of the notations see Table 1.

$p_1/p_2 = 2$, $M^2 = 2.5$

TABLE 22

ϕ, ψ METHOD, PHASE ERROR

$M^2 = 1.5$ $\alpha = 54.74°$ $\Delta x/\Delta y = .707$

PHASE ERROR IN DEGREES OVER ONE FULL WAVE IN x DIRECTION

θ DEGREES	p_1/p_2	$p_2 = 5°$	10°	15°	30°	45°	60°	90°
-35	153.236	287.85	320.93	333.26	346.29	350.79	353.06	355.36
-30	7.674	24.54	70.35	110.98	185.90	225.13	249.52	278.43
-25	3.905	7.12	25.32	48.41	112.08	155.12	183.99	219.96
-20	2.591	3.23	12.20	25.17	70.99	110.16	139.15	175.86
-15	1.915	1.79	6.93	14.81	46.94	79.66	106.63	142.35
-10	1.498	1.11	4.35	9.47	32.21	58.50	82.42	116.58
-5	1.212	.74	2.91	6.40	22.79	43.56	64.17	96.51
0	1.000	.51	2.03	4.49	16.51	32.83	50.28	80.75
5	.835	.37	1.45	3.23	12.18	24.99	39.62	68.26
10	.701	.27	1.06	2.37	9.09	19.17	31.40	58.30
15	.588	.20	.79	1.76	6.85	14.81	25.02	50.31
20	.490	.15	.59	1.31	5.19	11.49	20.05	43.90
25	.404	.11	.44	.99	3.96	8.97	16.18	38.78
30	.326	.08	.33	.74	3.03	7.05	13.19	34.74
35	.254	.06	.25	.56	2.34	5.61	10.92	31.63

For meaning of p_1 and p_2 see Table 1.

Aspect Ratio $\Delta x/\Delta y$ = const, chosen so that for $\theta = 0$ the mesh Courant Number is 1.

$M^2 = 1.5$

TABLE 23

φ, ψ METHOD, PHASE ERROR

$M^2 = 2.0$ $\alpha = 45.00°$ $\Delta x/\Delta y = 1.000$

PHASE ERROR IN DEGREES OVER ONE FULL WAVE IN x DIRECTION

θ DEGREES	P_1/P_2	$P_2 = 5°$	10°	15°	30°	45°	60°	90°
-40	11.430	44.43	106.56	151.07	223.01	258.02	278.79	302.23
-35	5.671	13.50	43.84	76.50	149.20	191.91	219.64	253.80
-30	3.732	6.12	22.07	42.88	103.18	146.00	175.56	213.36
-25	2.747	3.38	12.71	26.11	72.74	112.11	141.35	179.55
-20	2.145	2.07	7.98	16.91	52.07	86.35	113.96	151.10
-15	1.732	1.36	5.30	11.43	37.73	66.51	91.63	127.00
-10	1.428	.93	3.64	7.95	27.56	51.08	73.26	106.44
-5	1.192	.65	2.55	5.62	20.20	39.03	58.08	88.84
0	1.000	.46	1.81	4.00	14.78	29.56	45.53	73.74
5	.839	.32	1.28	2.84	10.72	22.10	35.19	60.82
10	.700	.22	.89	1.99	7.66	16.23	26.70	49.83
15	.577	.15	.61	1.36	5.33	11.61	19.81	40.60
20	.466	.10	.40	.89	3.57	8.02	14.28	32.97
25	.364	.06	.24	.55	2.25	5.28	9.97	26.85
30	.268	.03	.13	.30	1.30	3.27	6.74	22.16
35	.176	.01	.06	.14	.65	1.89	4.50	18.85
40	.087	.00	.02	.04	.28	1.08	3.18	16.88

For explanation of notations see Table 22.

$M^2 = 2$

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TABLE 24

φ, ψ METHOD, PHASE ERROR

$M^2 = 2.5$ $\alpha = 39.23°$ $\Delta x/\Delta y = 1.225$

PHASE ERROR IN DEGREES OVER ONE FULL WAVE IN x DIRECTION

θ DEGREES	ρ_1/ρ_2	$\rho_2 = 5°$	10°	15°	30°	45°	60°	90°
-50	91.308	249.89	297.47	316.46	337.26	344.62	348.37	352.18
-45	12.124	49.29	114.11	158.75	229.17	262.94	282.85	305.23
-40	6.440	17.34	53.73	90.00	164.39	205.90	232.35	264.46
-35	4.337	8.33	29.06	54.41	120.65	163.84	192.72	229.06
-30	3.230	4.73	17.42	34.78	89.45	131.13	160.59	198.23
-25	2.537	2.95	11.20	23.25	66.74	104.93	133.77	171.25
-20	2.057	1.96	7.55	16.05	49.99	83.63	110.95	147.48
-15	1.700	1.35	5.25	11.33	37.49	66.21	91.33	126.42
-10	1.420	.94	3.71	8.10	28.06	51.93	74.37	107.68
-5	1.192	.67	2.64	5.82	20.88	40.23	59.72	90.97
0	1.000	.47	1.88	4.16	15.36	30.66	47.13	76.11
5	.833	.33	1.32	2.94	11.10	22.87	36.41	62.99
10	.685	.23	.91	2.03	7.82	16.60	27.39	51.55
15	.551	.15	.60	1.35	5.31	11.62	19.96	41.79
20	.427	.09	.38	.85	3.42	7.77	14.04	33.75
25	.311	.05	.22	.50	2.06	4.93	9.55	27.48
30	.199	.03	.11	.26	1.14	2.99	6.45	23.03
35	.091	.01	.06	.13	.64	1.91	4.68	20.46
40	-.016	.01	.04	.10	.51	1.64	4.24	19.79
45	-.124	.02	.07	.16	.76	2.18	5.12	21.04
50	-.233	.04	.14	.33	1.40	3.54	7.32	24.20

For explanation of notations see Table 22.

$M^2 = 2.5$

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TABLE 25

ϕ, ψ METHOD, MODULUS

$M^2 = 1.5$ $\alpha = 54.74^\circ$ $p_1/p_2 = .500$

MODULUS AFTER ONE FULL WAVE

θ DEGREES	$\Delta x/\Delta y$	$p_2 = 5^\circ$	10°	15°	30°	45°	60°	90°
-35	.002	1.02	1.07	1.16	1.70	2.80	4.85	15.96
-30	.046	1.02	1.07	1.16	1.70	2.80	4.85	16.01
-25	.091	1.02	1.07	1.16	1.70	2.81	4.86	16.07
-20	.136	1.02	1.07	1.16	1.70	2.81	4.86	16.12
-15	.185	1.02	1.07	1.16	1.70	2.80	4.86	16.13
-10	.236	1.02	1.07	1.16	1.70	2.80	4.85	16.07
-5	.292	1.02	1.07	1.16	1.70	2.80	4.83	15.91
0	.354	1.02	1.07	1.16	1.70	2.79	4.81	15.67
5	.423	1.02	1.07	1.16	1.70	2.79	4.79	15.38
10	.505	1.02	1.07	1.16	1.70	2.78	4.76	15.07
15	.601	1.02	1.07	1.16	1.70	2.78	4.73	14.79
20	.721	1.02	1.07	1.16	1.70	2.77	4.71	14.55
25	.875	1.02	1.07	1.16	1.70	2.76	4.68	14.36
30	1.085	1.02	1.07	1.16	1.70	2.76	4.67	14.22
35	1.394	1.02	1.07	1.16	1.70	2.75	4.65	14.12

For explanation of notations see Table 1.

$p_1/p_2 = .5$, $M^2 = 1.5$

TABLE 26

ϕ, ψ METHOD, MODULUS

$M^2 = 2.0$ $\alpha = 45.00°$ $p_1/p_2 = .500$

MODULUS AFTER ONE FULL WAVE

θ DEGREES	$\Delta x/\Delta y$	$p_2 = 5°$	$10°$	$15°$	$30°$	$45°$	$60°$	$90°$
-40	.044	1.02	1.06	1.14	1.62	2.57	4.29	13.16
-35	.088	1.02	1.06	1.14	1.62	2.57	4.29	13.16
-30	.134	1.02	1.06	1.14	1.62	2.57	4.29	13.16
-25	.182	1.02	1.06	1.14	1.62	2.57	4.29	13.16
-20	.233	1.02	1.06	1.14	1.62	2.57	4.29	13.16
-15	.289	1.02	1.06	1.14	1.62	2.57	4.29	13.16
-10	.350	1.02	1.06	1.14	1.62	2.57	4.29	13.16
-5	.420	1.02	1.06	1.14	1.62	2.57	4.29	13.16
0	.500	1.02	1.06	1.14	1.62	2.57	4.29	13.16
5	.596	1.02	1.06	1.14	1.62	2.57	4.29	13.16
10	.714	1.02	1.06	1.14	1.62	2.57	4.29	13.16
15	.866	1.02	1.06	1.14	1.62	2.57	4.29	13.16
20	1.072	1.02	1.06	1.14	1.62	2.57	4.29	13.16
25	1.374	1.02	1.06	1.14	1.62	2.57	4.29	13.16
30	1.866	1.02	1.06	1.14	1.62	2.57	4.29	13.16
35	2.836	1.02	1.06	1.14	1.62	2.57	4.29	13.16
40	5.715	1.02	1.06	1.14	1.62	2.57	4.29	13.16

For explanation of the notations see Table 1.

$p_1/p_2 = .5,$ $M^2 = 2$

TABLE 27

Φ,ψ METHOD, MODULUS

M² = 2.5   α = 39.23°   p₁/p₂ = .500

$M^2 = 2.5$ $\alpha = 39.23°$ $p_1/p_2 = .500$

MODULUS AFTER ONE FULL WAVE

θ DEGREES	Δx/Δy	p₂ = 5°	10°	15°	30°	45°	60°	90°
-50	.007	1.02	1.07	1.15	1.65	2.64	4.47	14.05
-45	.051	1.02	1.07	1.15	1.65	2.64	4.47	14.05
-40	.095	1.02	1.07	1.15	1.65	2.64	4.47	14.05
-35	.141	1.02	1.07	1.15	1.65	2.64	4.47	14.06
-30	.190	1.02	1.07	1.15	1.65	2.64	4.47	14.07
-25	.241	1.02	1.07	1.15	1.65	2.64	4.47	14.09
-20	.298	1.02	1.07	1.15	1.65	2.64	4.47	14.10
-15	.360	1.02	1.07	1.15	1.65	2.64	4.47	14.10
-10	.431	1.02	1.07	1.15	1.65	2.64	4.47	14.08
-5	.514	1.02	1.07	1.15	1.65	2.64	4.47	14.04
0	.612	1.02	1.07	1.15	1.65	2.64	4.46	13.97
5	.735	1.02	1.07	1.15	1.65	2.64	4.45	13.89
10	.893	1.02	1.07	1.15	1.65	2.64	4.44	13.80
15	1.111	1.02	1.07	1.15	1.65	2.64	4.44	13.72
20	1.433	1.02	1.07	1.15	1.65	2.63	4.43	13.64
25	1.971	1.02	1.07	1.15	1.64	2.63	4.42	13.58
30	3.076	1.02	1.07	1.15	1.64	2.63	4.42	13.54
35	6.758	1.02	1.07	1.15	1.64	2.63	4.41	13.51
40	-37.276	1.02	1.07	1.15	1.64	2.63	4.41	13.49
45	-4.949	1.02	1.07	1.15	1.64	2.63	4.41	13.48
50	-2.629	1.02	1.07	1.15	1.64	2.63	4.41	13.49

For explanation of the notations see Table 1.

$p_1/p_2 = .5$, $M^2 = 2.5$

TABLE 28

ϕ, ψ METHOD, MODULUS

$M^2 = 1.5$ $\alpha = 54.74°$ $p_1/p_2 = 1.000$

MODULUS AFTER ONE FULL WAVE

θ DEGREES	$\Delta x/\Delta y$	$P_2 = 5°$	$10°$	$15°$	$30°$	$45°$	$60°$	$90°$
-35	.005	1.03	1.11	1.26	2.03	3.33	5.12	10.20
-30	.092	1.03	1.11	1.26	2.03	3.33	5.12	10.22
-25	.181	1.03	1.11	1.26	2.03	3.33	5.12	10.24
-20	.273	1.03	1.11	1.26	2.03	3.33	5.11	10.23
-15	.369	1.03	1.11	1.26	2.03	3.32	5.09	10.18
-10	.472	1.03	1.11	1.25	2.03	3.31	5.07	10.08
-5	.584	1.03	1.11	1.25	2.03	3.30	5.03	9.93
0	.707	1.03	1.11	1.25	2.02	3.29	4.99	9.74
5	.847	1.03	1.11	1.25	2.02	3.27	4.95	9.55
10	1.009	1.03	1.11	1.25	2.02	3.26	4.91	9.39
15	1.203	1.03	1.11	1.25	2.01	3.24	4.87	9.25
20	1.442	1.03	1.11	1.25	2.01	3.23	4.83	9.14
25	1.751	1.03	1.11	1.25	2.01	3.22	4.81	9.06
30	2.171	1.03	1.11	1.25	2.01	3.21	4.79	9.00
35	2.787	1.03	1.11	1.25	2.00	3.20	4.77	8.97

For explanation of the notations see Table 1.

$p_1/p_2 = 1,$ $M^2 = 1.5$

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TABLE 29

Φ, Ψ METHOD, MODULUS

$M^2 = 2.0$ $\alpha = 45.00°$ $p_1/p_2 = 1.000$

MODULUS AFTER ONE FULL WAVE

θ DEGREES	Δx/Δy	$p_2 = 5°$	10°	15°	30°	45°	60°	90°
-40	.087	1.03	1.10	1.23	1.91	3.02	4.53	8.68
-35	.176	1.03	1.10	1.23	1.91	3.02	4.53	8.68
-30	.268	1.03	1.10	1.23	1.91	3.02	4.53	8.68
-25	.364	1.03	1.10	1.23	1.91	3.02	4.53	8.68
-20	.466	1.03	1.10	1.23	1.91	3.02	4.53	8.68
-15	.577	1.03	1.10	1.23	1.91	3.02	4.53	8.68
-10	.700	1.03	1.10	1.23	1.91	3.02	4.53	8.68
-5	.839	1.03	1.10	1.23	1.91	3.02	4.53	8.68
0	1.000	1.03	1.10	1.23	1.91	3.02	4.53	8.68
5	1.192	1.03	1.10	1.23	1.91	3.02	4.53	8.68
10	1.428	1.03	1.10	1.23	1.91	3.02	4.53	8.68
15	1.732	1.03	1.10	1.23	1.91	3.02	4.53	8.68
20	2.145	1.03	1.10	1.23	1.91	3.02	4.53	8.68
25	2.747	1.03	1.10	1.23	1.91	3.02	4.53	8.68
30	3.732	1.03	1.10	1.23	1.91	3.02	4.53	8.68
35	5.671	1.03	1.10	1.23	1.91	3.02	4.53	8.68
40	11.430	1.03	1.10	1.23	1.91	3.02	4.53	8.68

For explanation of the notations see Table 1.

$p_1/p_2 = 1$, $M^2 = 2$

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TABLE 30

Φ,ψ METHOD, MODULUS

$M^2 = 2.5$ $\alpha = 39.23°$ $p_1/p_2 = 1.000$

MODULUS AFTER ONE FULL WAVE

θ DEGREES	Δx/Δy	$p_2 = 5°$	10°	15°	30°	45°	60°	90°
-50	.013	1.03	1.10	1.24	1.95	3.13	4.73	9.17
-45	.101	1.03	1.10	1.24	1.95	3.13	4.73	9.17
-40	.190	1.03	1.10	1.24	1.95	3.13	4.73	9.17
-35	.282	1.03	1.10	1.24	1.95	3.13	4.73	9.18
-30	.379	1.03	1.10	1.24	1.95	3.13	4.72	9.18
-25	.483	1.03	1.10	1.24	1.95	3.12	4.72	9.18
-20	.595	1.03	1.10	1.24	1.95	3.12	4.72	9.18
-15	.720	1.03	1.10	1.24	1.95	3.12	4.72	9.16
-10	.862	1.03	1.10	1.24	1.95	3.12	4.71	9.13
-5	1.027	1.03	1.10	1.24	1.95	3.12	4.70	9.09
0	1.225	1.03	1.10	1.24	1.95	3.11	4.69	9.03
5	1.470	1.03	1.10	1.24	1.94	3.11	4.67	8.98
10	1.787	1.03	1.10	1.24	1.94	3.10	4.66	8.92
15	2.222	1.03	1.10	1.24	1.94	3.10	4.65	8.88
20	2.867	1.03	1.10	1.24	1.94	3.10	4.64	8.84
25	3.943	1.03	1.10	1.24	1.94	3.09	4.63	8.82
30	6.153	1.03	1.10	1.24	1.94	3.09	4.62	8.80
35	13.516	1.03	1.10	1.24	1.94	3.09	4.62	8.79
40	-74.553	1.03	1.10	1.24	1.94	3.09	4.61	8.78
45	-9.899	1.03	1.10	1.24	1.94	3.09	4.61	8.78
50	-5.258	1.03	1.10	1.24	1.94	3.09	4.61	8.78

For explanation of the notations see Table 1.

$p_1/p_2 = 1$, $M^2 = 2.5$

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TABLE 31

Φ, Ψ METHOD, MODULUS

$M^2 = 1.5$ $\alpha = 54.74°$ $p_1/p_2 = 2.000$

MODULUS AFTER ONE FULL WAVE

θ DEGREES	Δx/Δy	$p_2 = 5°$	10°	15°	30°	45°	60°	90°
−35	.009	1.07	1.27	1.59	2.85	3.91	4.45	4.48
−30	.184	1.07	1.27	1.59	2.85	3.90	4.44	4.48
−25	.362	1.07	1.27	1.59	2.84	3.89	4.42	4.47
−20	.546	1.07	1.27	1.59	2.84	3.87	4.39	4.44
−15	.738	1.07	1.27	1.59	2.83	3.85	4.35	4.39
−10	.944	1.07	1.27	1.59	2.82	3.82	4.30	4.32
−5	1.167	1.07	1.27	1.58	2.81	3.79	4.25	4.24
0	1.414	1.07	1.27	1.58	2.80	3.75	4.20	4.18
5	1.694	1.07	1.27	1.58	2.78	3.72	4.15	4.13
10	2.019	1.07	1.27	1.58	2.77	3.69	4.11	4.09
15	2.406	1.07	1.27	1.58	2.76	3.67	4.09	4.07
20	2.885	1.07	1.27	1.58	2.75	3.65	4.06	4.05
25	3.501	1.07	1.27	1.58	2.74	3.63	4.05	4.04
30	4.341	1.07	1.27	1.58	2.73	3.62	4.03	4.03
35	5.575	1.07	1.27	1.58	2.73	3.61	4.03	4.02

For explanation of the notations see Table 1.

$p_1/p_2 = 2$, $M^2 = 1.5$

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TABLE 32

φ, ψ METHOD, MODULUS

$M^2 = 2.0$ $\alpha = 45.00°$ $p_1/p_2 = 2.000$

MODULUS AFTER ONE FULL WAVE

θ DEGREES	Δx/Δy	$p_2 = 5°$	10°	15°	30°	45°	60°	90°
-40	.175	1.06	1.24	1.52	2.62	3.53	3.99	4.00
-35	.353	1.06	1.24	1.52	2.62	3.53	3.99	4.00
-30	.536	1.06	1.24	1.52	2.62	3.53	3.99	4.00
-25	.728	1.06	1.24	1.52	2.62	3.53	3.99	4.00
-20	.933	1.06	1.24	1.52	2.62	3.53	3.99	4.00
-15	1.155	1.06	1.24	1.52	2.62	3.53	3.99	4.00
-10	1.400	1.06	1.24	1.52	2.62	3.53	3.99	4.00
-5	1.678	1.06	1.24	1.52	2.62	3.53	3.99	4.00
0	2.000	1.06	1.24	1.52	2.62	3.53	3.99	4.00
5	2.384	1.06	1.24	1.52	2.62	3.53	3.99	4.00
10	2.856	1.06	1.24	1.52	2.62	3.53	3.99	4.00
15	3.464	1.06	1.24	1.52	2.62	3.53	3.99	4.00
20	4.289	1.06	1.24	1.52	2.62	3.53	3.99	4.00
25	5.495	1.06	1.24	1.52	2.62	3.53	3.99	4.00
30	7.464	1.06	1.24	1.52	2.62	3.53	3.99	4.00
35	11.343	1.06	1.24	1.52	2.62	3.53	3.99	4.00
40	22.860	1.06	1.24	1.52	2.62	3.53	3.99	4.00

For explanation of the notations see Table 1.

$p_1/p_2 = 2$, $M^2 = 2$

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TABLE 33

φ, ψ METHOD, MODULUS

$M^2 = 2.5$ $\alpha = 39.23°$ $p_1/p_2 = 2.000$

MODULUS AFTER ONE FULL WAVE

θ DEGREES	Δx/Δy	$p_2 = 5°$	10°	15°	30°	45°	60°	90°
-50	.027	1.06	1.25	1.54	2.70	3.66	4.14	4.16
-45	.202	1.06	1.25	1.54	2.70	3.66	4.14	4.16
-40	.380	1.06	1.25	1.54	2.70	3.66	4.14	4.16
-35	.565	1.06	1.25	1.54	2.70	3.66	4.14	4.16
-30	.758	1.06	1.25	1.54	2.70	3.65	4.13	4.16
-25	.965	1.06	1.25	1.54	2.70	3.65	4.13	4.15
-20	1.191	1.06	1.25	1.54	2.70	3.65	4.12	4.14
-15	1.441	1.06	1.25	1.54	2.69	3.64	4.11	4.13
-10	1.724	1.06	1.25	1.54	2.69	3.63	4.09	4.10
-5	2.054	1.06	1.25	1.54	2.69	3.62	4.08	4.08
0	2.449	1.06	1.25	1.54	2.68	3.61	4.06	4.06
5	2.939	1.06	1.25	1.54	2.68	3.60	4.04	4.05
10	3.574	1.06	1.25	1.54	2.67	3.59	4.03	4.03
15	4.444	1.06	1.25	1.54	2.67	3.58	4.02	4.02
20	5.733	1.06	1.25	1.54	2.67	3.57	4.01	4.02
25	7.886	1.06	1.25	1.54	2.66	3.57	4.01	4.01
30	12.305	1.06	1.25	1.54	2.66	3.57	4.00	4.01
35	27.031	1.06	1.25	1.54	2.66	3.56	4.00	4.01
40	-149.106	1.06	1.25	1.54	2.66	3.56	4.00	4.01
45	-19.798	1.06	1.25	1.54	2.66	3.56	4.00	7.09
50	-10.516	1.06	1.25	1.54	2.66	3.56	4.00	4.01

For explanation of the notations see Table 1.

$p_1/p_2 = 2$, $M^2 = 2.5$

TABLE 34

Φ, ψ METHOD, MODULUS

$M^2 = 1.5$ $\alpha = 54.74°$ $\Delta x/\Delta y = .707$

MODULUS AFTER ONE FULL WAVE

θ DEGREES	p_1/p_2	$p_2 = 5°$	10°	15°	30°	45°	60°	90°
-35	153.236	1.21	1.06	1.02	1.01	1.00	1.00	1.00
-30	7.674	1.71	2.77	3.15	2.49	1.88	1.55	1.26
-25	3.905	1.22	1.77	2.41	3.35	3.13	2.68	1.99
-20	2.591	1.11	1.41	1.84	3.17	3.75	3.74	3.15
-15	1.915	1.07	1.26	1.55	2.77	3.85	4.45	4.65
-10	1.498	1.05	1.18	1.40	2.44	3.70	4.83	6.34
-5	1.212	1.03	1.14	1.31	2.20	3.49	4.97	8.08
0	1.000	1.03	1.11	1.25	2.02	3.29	4.99	9.74
5	.835	1.02	1.10	1.22	1.90	3.11	4.94	11.26
10	.701	1.02	1.08	1.19	1.81	2.97	4.87	12.59
15	.588	1.02	1.08	1.17	1.74	2.85	4.78	13.72
20	.490	1.02	1.07	1.16	1.69	2.76	4.70	14.66
25	.404	1.02	1.07	1.15	1.66	2.69	4.63	15.44
30	.326	1.02	1.06	1.14	1.63	2.63	4.57	16.07
35	.254	1.02	1.06	1.14	1.61	2.59	4.53	16.58

For meaning of p_1 and p_2 see Table 1.

Aspect Ratio $\Delta x/\Delta y$ = const, chosen so that for $\theta = 0$ the mesh Courant Number is 1.

$M^2 = 1.5$

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TABLE 35

Φ, ψ METHOD, MODULUS

$M^2 = 2.0$ $\alpha = 45.00°$ $\Delta x/\Delta y = 1.000$

MODULUS AFTER ONE FULL WAVE

θ DEGREES	p_1/p_2	$p_2 = 5°$	10°	15°	30°	45°	60°	90°
-40	11.430	2.13	2.87	2.66	1.76	1.39	1.23	1.11
-35	5.671	1.38	2.15	2.73	2.77	2.22	1.83	1.43
-30	3.732	1.18	1.64	2.19	3.10	2.97	2.58	1.96
-25	2.747	1.11	1.40	1.81	2.98	3.39	3.29	2.72
-20	2.145	1.07	1.27	1.57	2.71	3.53	3.85	3.67
-15	1.732	1.05	1.20	1.43	2.45	3.49	4.22	4.78
-10	1.428	1.04	1.15	1.33	2.22	3.36	4.44	6.02
-5	1.192	1.03	1.12	1.27	2.04	3.19	4.53	7.34
0	1.000	1.03	1.10	1.23	1.91	3.02	4.53	8.68
5	.839	1.02	1.09	1.19	1.80	2.87	4.49	10.00
10	.700	1.02	1.08	1.17	1.72	2.74	4.42	11.27
15	.577	1.02	1.07	1.15	1.65	2.63	4.34	12.43
20	.466	1.02	1.06	1.14	1.61	2.54	4.26	13.47
25	.364	1.01	1.06	1.13	1.57	2.47	4.19	14.35
30	.268	1.01	1.05	1.12	1.54	2.42	4.14	15.06
35	.176	1.01	1.05	1.12	1.52	2.38	4.09	15.57
40	.087	1.01	1.05	1.12	1.51	2.36	4.07	15.88

For explanation of the notations see Table 34.

$M^2 = 2$

TABLE 36

φ, ψ METHOD, MODULUS

$M^2 = 2.5$ $\alpha = 39.23^\circ$ $\Delta x / \Delta y = 1.225$

MODULUS AFTER ONE FULL WAVE

θ DEGREES	P_1/P_2	$P_2 = 5^\circ$	10°	15°	30°	45°	60°	90°
-50	91.308	1.49	1.14	1.06	1.02	1.01	1.00	1.00
-45	12.124	2.26	2.95	2.67	1.73	1.37	1.22	1.10
-40	6.440	1.49	2.39	2.92	2.66	2.07	1.70	1.35
-35	4.337	1.25	1.84	2.46	3.13	2.78	2.33	1.75
-30	3.230	1.15	1.53	2.04	3.17	3.30	3.00	2.33
-25	2.537	1.10	1.36	1.75	2.98	3.57	3.59	3.08
-20	2.057	1.07	1.26	1.56	2.73	3.65	4.06	4.00
-15	1.700	1.05	1.20	1.44	2.48	3.59	4.39	5.08
-10	1.420	1.04	1.16	1.35	2.27	3.45	4.59	6.30
-5	1.192	1.03	1.13	1.28	2.09	3.29	4.68	7.63
0	1.000	1.03	1.10	1.24	1.95	3.11	4.69	9.03
5	.833	1.02	1.09	1.20	1.83	2.95	4.64	10.46
10	.685	1.02	1.08	1.18	1.74	2.80	4.56	11.87
15	.551	1.02	1.07	1.16	1.67	2.68	4.47	13.19
20	.427	1.02	1.06	1.14	1.62	2.58	4.38	14.36
25	.311	1.01	1.06	1.13	1.58	2.50	4.30	15.33
30	.199	1.01	1.05	1.13	1.55	2.45	4.24	16.04
35	.091	1.01	1.05	1.12	1.54	2.42	4.21	16.46
40	-.016	1.01	1.05	1.12	1.53	2.41	4.20	16.55
45	-.124	1.01	1.05	1.12	1.54	2.42	4.21	16.32
50	-.233	1.01	1.06	1.13	1.56	2.46	4.25	15.77

For explanation of the notations see Table 34.

$M^2 = 2.5$

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